

A novel passive electric network analog to Kirchhoff-Love plate designed to efficiently damp forced vibrations by distributed piezoelectric transducers.

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ABSTRACT

Recently the concept of Piezo-Electro-Mechanical (PEM) structural member has been developed,^{2,1} Given a structural member, a set of piezoelectric actuators is uniformly distributed on it and electrically interconnected by one of its analog circuits. In this way it is obtained a high-performances piezoelectric structural-modification aiming to multimodal mechanical vibrations control.³ In the present paper it is addressed the problem of synthesizing an electrically dissipative PEM Kirchhoff-Love (K-L) plate by using completely passive electric networks.

1 Introduction

The problem of synthesizing a PEM K-L plate has been faced in this work discretizing the Lagrange functional governing the motion of K-L plate by a finite difference method, so determining a novel electric circuit governed by the obtained finite dimensional Lagrangian and subsequently suitably introducing optimal dissipative circuital elements. We design a realistic simply supported PEM K-L plate and study its performances to show its technical feasibility. This is obtained via the following steps: I) synthesizing an analog circuit of K-L plate constituted by *completely* passive elements; (in¹ the synthesized analog circuit cannot dispense with external feeding); II) modelling the Piezo-Electro-Mechanical plates (we want to design) by homogenized PDEs; III) establishing how to add suitable resistive elements in the previously found analog circuit to obtain a multimode (i.e. modal independent) damping of mechanical vibrations: the structural members thus obtained will be called *electrically-dissipative PEM plates*. Let us finally stress the fact that the novel circuits which we synthesize are passive, easily realizable and easily adaptable in the design of the realistic engineering devices apt to vibration and noise control.

2 Piezo-electro-mechanical K-L Plates

2.1 Analog circuit

In order to find an electric circuit analog to the K-L plate we start to consider the homogeneous model of the plate, the vibration of which is governed (see e.g.⁹) by the Lagrange functional where the deformation energy U^m and the kinetic one T^m are given by¹

$$U^m = \frac{S}{2} \int_{\Omega} \left((u_{,xx}^2 + u_{,yy}^2 + 2\nu u_{,xx} u_{,yy}) + 2(1-\nu) u_{,xy}^2 \right) dx dy$$

$$T^m = \frac{\rho h}{2} \int_{\Omega} \left(\frac{du}{dt} \right)^2.$$

The action time density for K-L plate (when ignoring dissipative phenomena) is therefore $\mathcal{L}^m = T^m - U^m$. The K-L plate is an infinite dimensional mathematical model. In order to synthesize one among its *lumped* circuital analogs we introduce a *finite* set of Lagrange parameters describing -in an approximate way- the state of the K-L plates: the set of displacements sampled at a fixed grid in the plate reference configuration. Let us label the generic point of the grid by the subscripts $(\cdot)_{i,j}$ so that the value at this point of the generic field f will be denoted by $f_{i,j}$. The step of the grid (assumed to be equal along both x and y directions) will be denoted by ε . Consistently the spatial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ will be approximated by the forward finite differences operators $\left(\frac{\tilde{\Delta}u}{\varepsilon}\right)_{i,j}, \left(\frac{\tilde{\Delta}v}{\varepsilon}\right)_{i,j}$ where $\left(\tilde{\Delta}f\right)_k = f_{k+1} - f_k$ stands for the simple difference operator. The introduction of the dimensionless deflection $\tilde{u} = u/u_o$ and the time derivative operator $d(\cdot)/d\tilde{t} = t_o d(\cdot)/dt =: (\cdot)$ by means of the characteristic deflection u_o and time t_o allows for writing the discrete deformation energy U_d^m and kinetic energy T_d^m of the K-L plate, in the following form:

$$U_d^m = \frac{S}{2} \left(\frac{u_o}{\varepsilon} \right)^2 \sum_i \sum_j \left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j}^2 + \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j}^2 + 2\nu \left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j} \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j} + 2(1-\nu) \left(\tilde{\Delta}_{xy} \tilde{u} \right)_{i,j}^2$$

$$T_d^m = \frac{\rho h}{2} \left(\frac{u_o \varepsilon}{t_o} \right)^2 \sum_i \sum_j \left(\dot{\tilde{u}} \right)_{i,j}^2 \quad (1)$$

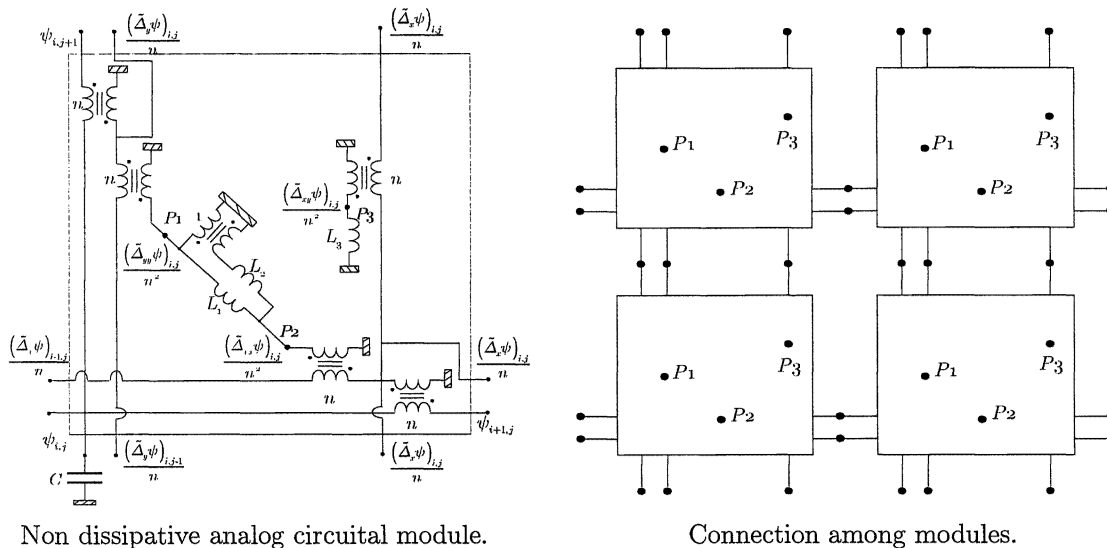
The approximating discrete Lagrangian system that we consider is close to the K-L plate model like the discrete differences are close to the derivatives: the actions considered in both models are likely to be sufficiently close when suitable regularity conditions on displacement fields are assumed. We leave to a more accurate mathematical treatment a precise statement of such regularity conditions and error estimates.

Given a quadratic finite dimensional Lagrange system and the corresponding Lagrange function, for instance $\mathcal{L}_d^m := T_d^m - U_d^m$, it is well known (see e.g.⁷) that it is possible to synthesize a lumped circuit governed by the same evolution (Euler-Lagrange) equations. This can be done, once the Lagrange parameters are recognized as the analogs of the flux-linkages of suitable nodes in the electric circuit, either identifying the deformation mechanical energy with magnetic energy (associated with inductors). We will choose the first option. Indeed to every node of the sampling grid previously introduced we will associate a principal node (denoted $n_{i,j}$) of the analog circuit: the time-integral $\psi_{i,j}$ of its potential drop with respect to ground will correspond to the deflection $u_{i,j}$. Hence, connecting every node to ground by means of capacitors, it is determined a tree for the analog circuit in which the capacitive electric energy T^e will parallel the mechanical kinetic energy T_d^m . Consequently the elastic energy U_d^m of the plate may be paralleled by the magnetic energy U^e stored into the circuital co-tree constituted by set of

¹The meanings of the introduced symbols are: K-L Plate bending stiffness: $S = \frac{h^3 E}{12(1-\nu^2)}$; thickness: h ; Young Modulus: E ; Poisson ratio: ν ; plate domain: Ω ; vertical deflection: u ; volumetric mass density ρ ; time variable: t . Moreover we have introduced a Cartesian system of coordinates (x, y) on the plate reference configuration Ω and denoted by $(\cdot)_{,x}, (\cdot)_{,y}$ the derivatives with respect to x, y variables respectively.

inductors opportunely connected to the circuitual tree by means of an appropriate network of electric transformers. As known the transformers do not store energy, they simply connect the nodes $n_{i,j}$ to some suitable extra auxiliary ones. The flux-linkages of these auxiliary nodes are expressed in terms of the flux voltages $\psi_{i,j}$ exactly by means of those finite differences determining the dependence of U_d^m on the deflection $u_{i,j}$.

Because of the made analogy choice, the electric circuit synthesized will be coupled to the plate only in dynamic conditions: it is an open problem to determine a static analog for K-L plate. In figure it is presented one of the possible topologies for the circuitual module (corresponding to the i,j node) able to realize an electric analog for the K-L plate and the connection among different modules:



Non dissipative analog circuitual module.

Connection among modules.

The searched electromechanical analogy will therefore find its quantitative expression when one will have introduced the dimensionless flux-linkage $\psi = \psi/\psi_o$ by means of a characteristic flux linkage ψ_o . Consequently we can express the electric energies U^e, T^e in the following form:

$$\begin{aligned}
 U^e = & \frac{1}{2n^4} \frac{L_1 + L_2}{L_1 L_2} \psi_o^2 \sum_i \sum_j \left(\tilde{\Delta}_{xx} \tilde{\psi} \right)_{i,j}^2 + \left(\tilde{\Delta}_{yy} \tilde{\psi} \right)_{i,j}^2 + 2 \frac{L_1 - L_2}{L_1 + L_2} \left(\tilde{\Delta}_{xx} \tilde{\psi} \right)_{i,j} \left(\tilde{\Delta}_{yy} \tilde{\psi} \right)_{i,j} + \\
 & + \frac{L_2 L_1}{L_3 (L_1 + L_2)} \left(\tilde{\Delta}_{xy} \tilde{\psi} \right)_{i,j}^2 \\
 T^e = & \frac{C \psi_o^2}{2t_o^2} \sum_i \sum_j \left(\dot{\tilde{\psi}} \right)_{i,j}^2 .
 \end{aligned}
 \tag{2}$$

The electric analog circuit found allow for quantitative analogies with the K-L plate once the coefficients of the electric-energies (see 2) are assumed to be equal to the coefficients of the mechanic ones (see 1). Simple algebra allows us to get the following equivalent equalities in which the electric impedances needed in the analog circuit are explicitly given in terms of mechanical parameters, scaling factors and transformers turn-ratios:

$$\begin{aligned}
 n^4 L_1 = & \frac{2}{S(1-\nu)} \left(\frac{\varepsilon \psi_o}{u_o} \right)^2 , \\
 n^4 L_2 = & \frac{2}{S(1+\nu)} \left(\frac{\varepsilon \psi_o}{u_o} \right)^2 , \\
 n^4 L_3 = & \frac{1}{2S(1-\nu)} \left(\frac{\varepsilon \psi_o}{u_o} \right)^2 , \\
 C = & \rho h \left(\frac{u_o \varepsilon}{\psi_o} \right)^2 .
 \end{aligned}
 \tag{3}$$

It is evident from the four equation (3) that the choice of the circuital parameters L_1, L_2, L_3, n and C is not unique. Indeed the introduction of transformers in the circuital analogs, not only allows for the synthesis of completely passive networks (which has not been possible in¹) but supplies also additional degrees of freedom (turns ratios) which can be chosen so to decrease the high inductances values usually requested in the electric circuits driving piezoelectric transducers.

2.2 Piezoelectric actuator modelling

In the analog (lumped) circuit which has been synthesized in the previous subsection some circuital elements are capacitors. They connect each electric node i, j to the ground. Therefore in the analog electric circuit it is easy to recognize a completely capacitive (i.e. constituted only by capacitive branches) tree: its co-tree being purely inductive (i.e. constituted by inductances and ideal transformers).

By making use of the so found analog circuit, the piezo-electro-mechanical (PEM) plate is easily conceived by connecting the previously described inductive co-tree in the electric analog network to the electric terminals of an array of distinct piezoelectric transducers uniformly distributed upon the plate surfaces (see figure 1): indeed the electric Norton equivalent for any of such transducer is given by a capacitor in parallel connection with a current generator. We underline here that the transducers are bonded to the host plate and that this bonding assures their mechanical interaction with the plate itself: moreover their piezoelectric transverse isotropy axes will be oriented coherently. In (4) it is investigated the theoretical reason for which -in order to control, in the most efficient way, the mechanical vibrations of a structural member using an array of piezoelectric transducers- one must employ an analog network. We limit ourselves here to state that an analog circuit is able to resonate at every proper frequency of the given structural member showing exactly the same spatial modal forms and therefore it is able to optimize the efficiency of the chosen energy transduction.

In the present section it is shown that one has to modify the Lagrangians previously introduced in order to model the behavior of the array of piezoelectric transducers adding their contribution to the mechanical deformation and kinetic energies and to the electric inductive and capacitive energies.

Moreover all addends in the Lagrangian for the PEM plate needed to model the piezoelectric transducers will be expressed in terms of those constitutive parameters usually introduced to characterize their response.

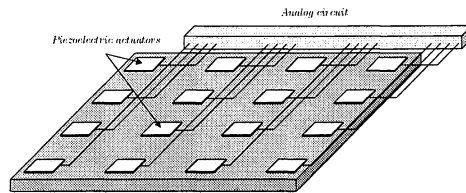


Figure 1: Pem Plate

In order to find the piezoelectric elastic deformation energy-contribute and the electric capacitive one, let us consider the general three-dimensional constitutive relations for a transversely-isotropic material (see⁸), that reduce in the case of plane stress (along xy directions) and electric field non-vanishing only in the direction z of transverse isotropy to

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ D_z \end{bmatrix} = \begin{bmatrix} \frac{E_p}{1-\nu_p^2} & \frac{E_p \nu_p}{1-\nu_p^2} & 0 & -d_{31} \frac{E_p}{1-\nu_p} \\ \frac{E_p \nu_p}{1-\nu_p^2} & \frac{E_p}{1-\nu_p^2} & 0 & -d_{31} \frac{E_p}{1-\nu_p} \\ 0 & 0 & \frac{E_p}{2(1+\nu_p)} & 0 \\ d_{31} \frac{E_p}{1-\nu_p} & d_{31} \frac{E_p}{1-\nu_p} & 0 & \left(\varepsilon_T - \frac{2d_{31}^2 E_p}{1-\nu_p} \right) \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ \gamma_{xy} \\ E_z \end{bmatrix} \quad (4)$$

which relates the electric displacement D_z and stress components $\sigma_x, \sigma_y, \sigma_{xy}$ to the mechanical deformations e_x, e_y, γ_{xy} , and the electric field E_z . The quantities ν_p, E_p , denote the Poisson coefficient and the elastic modulus of the considered piezoelectric material measured when $E_z = 0$, while d_{31} and ε_T represent the transverse piezoelectric coupling coefficient and the dielectric constant measured when the mechanical stresses vanish.

From an electric point of view the transducers will play the role of the capacitors and of a mechanical deformation-rate driven current generator. Moreover we will assume that inside every patch (once modelled as a three-dimensional continuum) the fields are those characteristic of plane stress and therefore independent of the z coordinate. Finally we assume that every sampling node introduced in the previous section is the geometrical center of exactly one transducer, consequently labelled by the same pair of indices.

Therefore it is easy to relate the kinematics of the patch i, j to the kinematics of the host plate getting

$$e_x = -\frac{h}{2}u_{xx}H_{i,j}; \quad e_y = -\frac{h}{2}u_{yy}H_{i,j}; \quad \gamma_{xy} = -hu_{xy}H_{i,j}; \quad E_z = \frac{V_{i,j}}{\delta}H_{i,j} \quad (5)$$

where $V_{i,j}$ is the potential drop between the transducers terminals which will be identified with the voltage drop $\dot{\psi}_{i,j}$ of the i, j node with respect to ground and we introduced the following Heaviside function:

$$H_{i,j} = \begin{cases} 1 & , |x - x_i| \leq l_p/2, \quad \text{and} \quad |y - y_j| \leq l_p/2 \\ 0 & , \quad \text{elsewhere} \end{cases} .$$

We explicitly remark that under the accepted assumptions inside every single transducer the electric field E_z is simultaneously independent of x, y and z variables. As a consequence, the capacity C of the piezoelectric transducer is given by $C = \left(\varepsilon_T - \frac{2d_{31}^2 E_p}{1-\nu_p} \right) \frac{l_p^2}{\delta}$

On the other hand the elastic energy contributes $(U_{mp})_{i,j}$ and the electric one $(T_{ep})_{i,j}$ due to the i, j transducer to the Lagrangian of the electromechanical system can be estimated as follows:

$$\begin{aligned} (U_{mp})_{i,j} &= \frac{1}{2} \int_{\Omega} H_{i,j} \left\{ \int_{\frac{h}{2}}^{\frac{h}{2}+\delta} \sigma_x e_x + \sigma_y e_y + \sigma_{xy} \gamma_{xy} \right\} = \\ &= \int_{\Omega} H_{i,j} \left\{ \frac{\delta h^2 E_p}{8(1-\nu_p^2)} [(u_{xx}^2 + u_{yy}^2 + 2\nu_p u_{xx} u_{yy}) + 2(1-\nu_p) u_{xy}^2] + d_{31} \frac{h E_p}{4(1-\nu_p)} (u_{xx} + u_{yy}) \dot{\psi}_{i,j} \right\} = \\ (T_{ep})_{i,j} &= \frac{1}{2} \int_{\Omega} H_{i,j} \left\{ \int_{\frac{h}{2}}^{\frac{h}{2}+\delta} D_z E_z \right\} = \int_{\Omega} H_{i,j} \left\{ \frac{1}{\delta} \left(\frac{\varepsilon_T}{2} - \frac{E_p}{1-\nu_p} d_{31}^2 \right) \dot{\psi}_{i,j}^2 - d_{31} \frac{h E_p}{4(1-\nu_p)} (u_{xx} + u_{yy}) \dot{\psi}_{i,j} \right\} \end{aligned}$$

While the kinetic energy contribute due to the piezo-layer to the total kinetic energy of the system is:

$$(T_{mp})_{i,j} = \frac{1}{2} \delta \int_{\Omega} H_{i,j} \rho_p \dot{u}^2,$$

where ρ_p denotes the volumetric density of mass for the piezoelectric material

In conclusion, the total contributes to the Lagrangian of the system given by the entire array of piezoelectric transducers when expressed in terms of non-dimensional discrete variables $(\tilde{u}_{i,j}, \dot{\tilde{\psi}}_{i,j})$ become:

$$\begin{aligned} U_p^m &= \eta^2 \frac{\delta h^2 E_p}{8(1-\nu_p^2)} \left(\frac{u_0}{\varepsilon} \right)^2 \sum_{ij} \left((\tilde{\Delta}_{xx} \tilde{u})_{i,j}^2 + (\tilde{\Delta}_{yy} \tilde{u})_{i,j}^2 + 2\nu_p (\tilde{\Delta}_{xx} \tilde{u})_{i,j} (\tilde{\Delta}_{yy} \tilde{u})_{i,j} + 2(1-\nu_p) (\tilde{\Delta}_{xy} \tilde{u})_{i,j}^2 \right) + \\ &+ \eta^2 \frac{h E_p}{4(1-\nu_p)} \frac{u_0 \psi_0}{t_0} d_{31} \sum_{ij} \dot{\tilde{\psi}}_{i,j} \left((\tilde{\Delta}_{xx} \tilde{u})_{i,j} + (\tilde{\Delta}_{yy} \tilde{u})_{i,j} \right) \\ T_p^m &= \eta^2 \frac{\rho_p \delta}{2} \left(\frac{u_0 \varepsilon}{t_0} \right)^2 \sum_{ij} \dot{\tilde{u}}_{i,j}^2 \\ T_p^e &= \frac{1}{\delta} \left(\frac{\varepsilon_T}{2} - \frac{E_p}{1-\nu_p} d_{31}^2 \right) \sum_{ij} \dot{\tilde{\psi}}_{i,j}^2 - \eta^2 \frac{h E_p}{4(1-\nu_p)} \frac{u_0 \psi_0}{t_0} d_{31} \sum_{ij} \dot{\tilde{\psi}}_{i,j} \left((\tilde{\Delta}_{xx} \tilde{u})_{i,j} + (\tilde{\Delta}_{yy} \tilde{u})_{i,j} \right) \end{aligned}$$

where η^2 is the ratio between the area covered by the array of piezoelectric transducers and that of the plate surface.

In order to be tuned with the mechanical vibrations the analog circuit have to be designed so to take into account the additional stiffness and mass due to the presence of the piezo-array.

2.3 PEM model

In conclusion the finite dimensional model describing the evolution of the piezo-electro-mechanical K-L plate is a Lagrangian system the Lagrangian parameters of which are the nodal displacements $\tilde{u}_{i,j}$ and the nodal flux-linkages $\tilde{\psi}_{i,j}$. The potential energy and the kinetic energy of the system are given by

$$U = \left(\frac{S}{2} \left(\frac{u_n}{\varepsilon} \right)^2 + 2\eta^2 \frac{\delta h^2 E_p}{8(1-\nu_p^2)} \left(\frac{u_n}{\varepsilon} \right)^2 \right) \sum_{ij} \left\{ \left(\left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j}^2 + \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j}^2 + 2\nu_t \left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j} \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j} \right) + \right. \\ \left. + \left(\left(\tilde{\Delta}_{xx} \tilde{\psi} \right)_{i,j}^2 + \left(\tilde{\Delta}_{yy} \tilde{\psi} \right)_{i,j}^2 + 2\nu_t \left(\tilde{\Delta}_{xx} \tilde{\psi} \right)_{i,j} \left(\tilde{\Delta}_{yy} \tilde{\psi} \right)_{i,j} \right) + \right. \\ \left. + 2(1-\nu_t) \left(\left(\tilde{\Delta}_{xy} \tilde{u} \right)_{i,j}^2 + \left(\tilde{\Delta}_{xy} \tilde{\psi} \right)_{i,j}^2 \right) \right\} + \\ + 2\mathcal{E}_{op}^c \sum_{ij} \dot{\tilde{\psi}}_{i,j} \left(\left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j} + \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j} \right), \quad (6)$$

$$T = \left(\frac{\rho h}{2} \left(\frac{u_o \varepsilon}{t_o} \right)^2 + \rho h \left(\frac{u_o \varepsilon}{t_o} \right)^2 \right) \sum_{ij} \left(\dot{\tilde{u}}_{i,j}^2 + \dot{\tilde{\psi}}_{i,j}^2 \right) - 2\mathcal{E}_{op}^c \sum_{ij} \dot{\tilde{\psi}}_{i,j} \left(\left(\tilde{\Delta}_{xx} \tilde{u} \right)_{i,j} + \left(\tilde{\Delta}_{yy} \tilde{u} \right)_{i,j} \right), \quad (7)$$

where we introduced the piezo-electro-mechanical coupling term $\mathcal{E}_{op}^c = \eta^2 \frac{h E_p}{4(1-\nu_p)} \frac{u_o \psi_o}{t_o} d_{31}$.

From the formulas 6, 7 constituting the Lagrangian of the system, it is easy to derive the following set of discrete Euler-Lagrange equations for the PEM plate²:

$$\left(\tilde{\Delta}^4 \tilde{u} \right)_{i,j} + \frac{T_o}{U_o} \ddot{\tilde{u}}_{i,j} + 2 \frac{\mathcal{E}_{op}^c}{U_o} \left(\tilde{\Delta}^2 \dot{\tilde{\psi}} \right)_{i,j} = 0 \\ \left(\tilde{\Delta}^4 \tilde{\psi} \right)_{i,j} + \frac{T_o}{U_o} \ddot{\tilde{\psi}}_{i,j} - 2 \frac{\mathcal{E}_{op}^c}{U_o} \left(\tilde{\Delta}^2 \dot{\tilde{u}} \right)_{i,j} = 0 \quad (8)$$

The coupled system of ODEs we have just obtained can have a huge number of unknown time functions. In order to get some qualitative informations about its solutions in some circumstances of interest in engineering applications one can consider the limit for the number of transducers tending to infinity. Such a formal procedure may have some direct applicative interest in vibration control of mechanical modes the wavelengths of which are greater than the size of the typical transducer: indeed in this case the homogenized model obtained after performing aforementioned formal limit may be useful also for quantitative prevision of the behavior of PEM plates. We believe that the rigorous mathematical proof given for a similar problem in,⁵ may be adapted to the case at hand. However the homogenized model may also be of use in a preliminary dimensioning and/or as a first step in an iterative modelling procedure aiming to account more precisely for the discrete lumped nature of analog circuits.

The idea of using a homogenized model instead of a discrete one is often used both in mechanics and the theory of circuits: indeed a truss modular structure and the lumped realization of a telegraphist transmission

² It has been introduced the following discrete difference operators $\tilde{\Delta}^2 = \tilde{\Delta}_{xx} + \tilde{\Delta}_{yy}$; $\tilde{\Delta}^4 = \tilde{\Delta}_{xx} \left(\tilde{\Delta}_{xx} \right) + 2\tilde{\Delta}_{yy} \left(\tilde{\Delta}_{xx} \right) + \tilde{\Delta}_{yy} \left(\tilde{\Delta}_{yy} \right)$.

line, which are naturally described by finite dimensional Lagrangian systems, are effectively studied using PDEs obtained by suitable homogenization procedures. It is reasonable to assume that the homogenized model of the system of ODEs (8), when the array is spatially uniform and constituted by transducers having the same size and electromechanical properties, is represented by the following system of PDEs ³

$$\begin{aligned}\tilde{\nabla}^4 \tilde{u} + \alpha^4 \ddot{\tilde{u}} - \beta^2 \tilde{\nabla}^2 \dot{\tilde{\psi}} &= 0 \\ \tilde{\nabla}^4 \tilde{\psi} + \alpha^4 \ddot{\tilde{\psi}} + \beta^2 \tilde{\nabla}^2 \dot{\tilde{u}} &= 0\end{aligned}\tag{9}$$

We recall that the K-L PEM plates are defined by tuning the electric network to get self-resonance of electric oscillations and mechanical vibrations. However the piezoelectric transducers exhibit some performances which are described by the piezoelectric constitutive parameters and represent the technological bottleneck limiting the efficiency of considered devices: in particular their capacity C (which has been found in terms of piezo-electro-mechanical parameters) must be considered as the datum which relates the dimensionalizing parameters u_o and ψ_o . Therefore it is needed now to find the expression for β in terms of the piezo-electro-mechanical constitutive parameters of the considered device. We get:

$$\beta^2 = -12d_{31} \frac{l_o^2 \eta}{t_o h} \sqrt{\frac{\delta E_p (1 - \nu^2)}{h E (1 - \nu_p)}} \sqrt{\frac{\rho}{\epsilon_T - \frac{2d_{31}^2 E_p}{1 - \nu_p}}} \frac{1}{1 + 6\eta^2 \frac{\delta E_p}{h E} \frac{1 - \nu^2}{1 - \nu_p^2}}\tag{10}$$

It is desirable to find a circuit resonating exactly at the same frequencies as the analog circuit and able to efficiently dissipate the energy content of every oscillation eigenmode: we prove in this section that such a circuit is obtained suitably adding to the analog circuit some dissipative branches between every i, j node and all its adjacent ones. Taking into account the results found in³ we are lead to this result by inserting the resistors in the co-tree of the analog indicated in the non dissipative circuit so to obtain, in the homogenized version of its evolution equations, a differential operator governing the dissipation having the same spectrum as to the one modelling the piezoelectric coupling and appearing in equations (9).

Using a synthesis procedure similar to the one used to obtain the analog non dissipative circuit we get the circuit sketched in figure 2).

so that the Euler-Lagrange electric equations of the dissipative system become:

$$\tilde{\nabla}^4 \tilde{u} + \alpha^4 \ddot{\tilde{u}} - \beta^2 \tilde{\nabla}^2 \dot{\tilde{\psi}} = 0\tag{11}$$

$$\tilde{\nabla}^4 \tilde{\psi} + \alpha^4 \ddot{\tilde{\psi}} + \beta^2 \tilde{\nabla}^2 \dot{\tilde{u}} - \gamma^2 \tilde{\nabla}^2 \dot{\tilde{\psi}} = 0\tag{12}$$

where the dimensionless dissipative coefficient γ^2 is given by:

$$\gamma^2 = \frac{1}{SR} \frac{l_o^2}{t_o} \frac{\delta h \rho}{\eta^2 \left(\epsilon_T - \frac{2d_{31}^2 E_p}{1 - \nu_p} \right)} \frac{1}{1 + 6\eta^2 \frac{\delta E_p}{h E} \frac{1 - \nu^2}{1 - \nu_p^2}}$$

³where the operator $\tilde{\nabla}$ represents the gradient in the previously introduced dimensionless space variables: $\tilde{\nabla} = \lim_{\tilde{\epsilon} \rightarrow 0} \frac{\tilde{\Delta}}{\tilde{\epsilon}} =$

$$l_o \lim_{\tilde{\epsilon} \rightarrow 0} \tilde{\Delta} \text{ and } \alpha^4 = \frac{T_o}{U_o \tilde{\epsilon}^4} = \frac{T_o^m}{\tilde{\epsilon}^4 U_o^m} \left(\frac{1 + \frac{2\gamma_{op}^m}{T_o^m}}{1 + \frac{2U_{op}^m}{U_o^m}} \right) = \frac{\rho h}{S} \frac{l_o^4}{t_o^2} \left(\frac{1 + 2\eta^2 \frac{\rho_p \delta}{\rho h}}{1 + 6\eta^2 \frac{\delta E_p}{h E} \frac{1 - \nu^2}{1 - \nu_p^2}} \right),$$

$$\beta^2 = -\frac{\epsilon_{op}^c}{U_o \tilde{\epsilon}^2} = -\frac{2\epsilon_{op}^c}{\tilde{\epsilon}^2 U_o^m} \frac{1}{1 + \frac{2U_{op}^m}{U_o^m}} = -12\eta^2 d_{31} \frac{E_p (1 - \nu^2)}{h^2 E (1 - \nu_p)} \frac{l_o^2 \psi_o}{u_o t_o} \frac{1}{1 + 6\eta^2 \frac{\delta E_p}{h E} \frac{1 - \nu^2}{1 - \nu_p^2}}$$

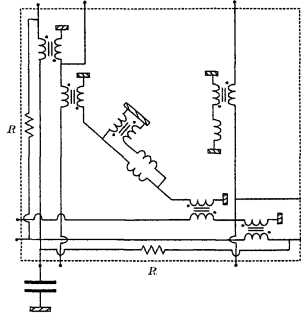


Figure 2: Dissipative analog module

In the language of the theory of control we introduce, via the array of piezoelectric transducers, an electric passive controller of mechanical vibrations: the constitutive parameter characterizing such a controller is γ^2 (for more details see²). We will determine the optimal value for γ^2 by means of the criterion of critical damping which is based on the so called pole placement technique.

2.4 Performances of the PEM plate

In the present section, the analysis of the PEM plate evolution, in the cases of free and forced vibrations, will be developed using Galerkin method. The electromechanical configuration fields $(\tilde{u}(x, y, t), \tilde{\psi}(x, y, t))$ are expressed as follows:

$$\begin{aligned}\tilde{u}(x, y, t) &= \sum_{\lambda} p_{\lambda}(t) m_{\lambda}(x, y) \\ \tilde{\psi}(x, y, t) &= \sum_{\lambda} q_{\lambda}(t) m_{\lambda}(x, y)\end{aligned}\tag{13}$$

where $p_{\lambda}(t)$ and $q_{\lambda}(t)$ are the time depending Fourier coefficients of the eigenfunctions $m_{\lambda}(x, y)$ coming from the eigenvalues problem

$$\tilde{\nabla}^4 m_{\lambda} = \lambda^4 m_{\lambda}$$

with suitable boundary conditions depending on the particular problem under consideration.

Introducing the previous expressions into the (11, 12) and projecting on the given basis, it is easy to derive the ODEs for the electrical and mechanical Fourier coefficients $p_{\lambda}(t)$, $q_{\lambda}(t)$

$$\begin{aligned}\lambda^4 p_{\lambda} + \alpha^4 \ddot{p}_{\lambda} - \beta^2 \sum_k A_{k\lambda} \dot{q}_k &= 0, \\ \lambda^4 q_{\lambda} + \alpha^4 \ddot{q}_{\lambda} + \beta^2 \sum_k A_{k\lambda} \dot{p}_k - \gamma^2 \sum_k A_{k\lambda} \dot{q}_k &= 0,\end{aligned}\tag{14}$$

where the coupling coefficients $A_{k\lambda}$ is given by ⁴

$$A_{k\lambda} = \langle \tilde{\nabla}^2 m_{\lambda}, m_k \rangle .$$

We underline that, in order to keep a complete electromechanical analogy, the boundary conditions for the electric field must be the same as those chosen for the mechanical field: this implies that the branches of the

⁴The operator \langle, \rangle represents the usual scalar product in the L^2 space.

analog circuit which are connected to the boundary nodes (i.e. to the terminals of the piezoelectric transducers close to the constrained edges of the PEM plate) must be suitably terminated.

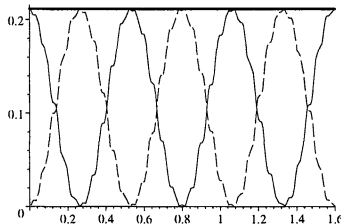
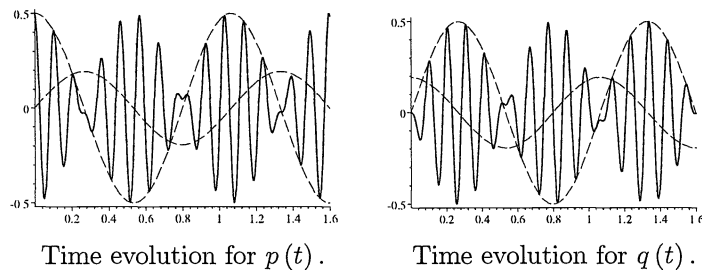
In order to show the electro-mechanical coupling in such a system we consider the free evolution of a simply supported non dissipative PEM plate presenting only an initial deformation on a given mode λ i.e. $p_\lambda(t)|_{t=0} = p_o$; the unique non-vanishing Fourier coefficients are:

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}_\lambda = \begin{bmatrix} p_o \left(\cos \omega_c t \cos \omega_m t + \frac{\omega_m}{\omega_c} \sin \omega_c t \sin \omega_m t \right) \\ p_o \left(\cos \omega_c t \sin \omega_m t - \frac{\omega_m}{\omega_c} \sin \omega_c t \cos \omega_m t \right) \end{bmatrix}.$$

where we defined the carrier and the modulating frequencies ω_c and ω_m :

$$\begin{aligned} \omega_c &= \frac{\omega_2 + \omega_1}{2} = \frac{\lambda^2}{\alpha^4} \frac{1}{2} \sqrt{\beta^4 + 4\alpha^4} \\ \omega_m &= \frac{\omega_2 - \omega_1}{2} = \frac{1}{2} \lambda^2 \frac{\beta^2}{\alpha^4} \end{aligned} \tag{15}$$

In the following figure and it is shown the evolution of $p(t)$ and $q(t)$ and the total mechanical and electrical energies relative to the plate mode (1, 1) for an aluminum plate having a 25% of its surface covered by commercial piezolayer and initially deformed on the same mode (1, 1):



Thick line: total energy; Solid line: total mechanical energy;
Dashed line: total electric energy.

Let us remark that the evolution shows a beating/amplitude modulation phenomenon between the electrical and mechanical subsystems: using the language of the theory of signals we can say that the effect of piezoelectric coupling can be regarded as an amplitude modulation of the mechanical signal in which the carrier frequency ω_c is close (when β is small) to that of the uncoupled mechanical system and the modulating frequency ω_m is proportional to the coupling coefficient (see definitions 15).

2.4.1 Optimal condition for the dissipative system

Once the optimal coupling between the mechanical system and its electric analog has been established, an optimal electric dissipation of the mechanical energy has to be obtained. We design an electrical feed-back controller to damp the mechanical vibrations by the optimally criterion requiring the double multiplicity of system eigenfrequencies based on the *pole placement technique*, well-established by the theory of control. In the theory of mechanical vibrations the same criterion is often used, but it is called the *criterion of critical damping*, stressing more specifically on the time decay properties of the corresponding time evolutions. More precisely: when the eigenvalues (i.e. the poles) of an evolution equation of the type have an algebraic multiplicity greater than one then the corresponding eigenvectors (i.e. the vibration modes) have the steepest exponential decay (for more details see² where the PEM beam is addressed).

A convenient positioning of the resistors in the analog circuit has already been established in the previous section in order to assure the capability of the electric feed-back to control mechanical vibrations independently of the modes. The criterion we will use simply aims to assign their optimal resistances value (or equivalently to find the dissipative electric coefficient γ) by imposing a critical damping condition for the entire electromechanical system. The condition which we choose exploits the *pole placement* technique. The poles \bar{s}_i of the electromechanical system see 14 can be regarded as the roots of the characteristic polynomial $P(s)$ ⁵:

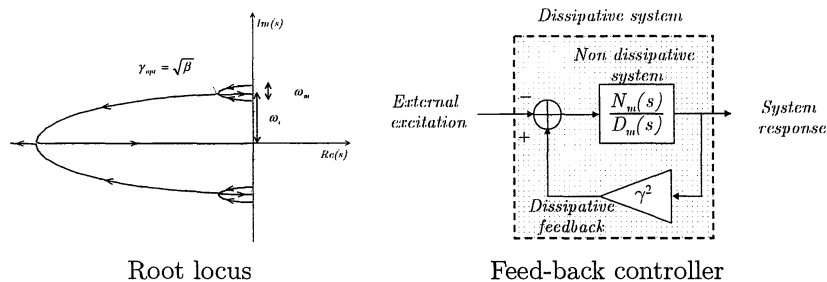
$$P(s) = D(s) + \gamma^2 N(s)$$

where

$$N(s) = s^4 + \frac{\lambda^4}{\alpha^8} (2\alpha^4 + \beta^4) s^2 + \frac{\lambda^8}{\alpha^8},$$

$$D(s) = s (s^2 \alpha^4 + \lambda^4) \frac{\lambda^2}{\alpha^8}.$$

which suggests to consider the dissipative contribute to the characteristic polynomial as a contribute arising because of the presence of a feed-back controller (see the following figures). Such a controller is able to displace the poles of the undamped system (which belong to the imaginary axis on the complex plane) back into the negative half-plane (see the following figures). It can be proven (see² for a detailed discussion of this topic) that the critical damping for the system is reached when the real part of the nearest-to-the-imaginary-axis poles is maximum. This is obviously equivalent to require that the poles nearest to the imaginary axis must be as far as possible from it. It is well known that this optimally condition (see²) is equivalent to require that the characteristic polynomial admits exactly two roots each of multiplicity two.



The chosen optimal criterion gives the following condition for the dissipation coefficient:

$$\gamma = \sqrt{2}\beta \tag{16}$$

It has to be explicitly remarked again that, as expected, the obtained value for the damping resistances actually is independent of the considered mode: the considerations developed in the previous section. are confirmed by the present analysis.

⁵The symbol s represent the Laplace variable while \mathbb{C} represent the complex field.

In order to show the mechanical damping of the so obtained PEM plate let us show the evolution of $p(t)$ and $q(t)$ relative to the plate mode (1, 1) for the aforementioned plate having an initial deformation velocity \dot{p}_o on the mode (1, 1):

$$\begin{aligned} p_{\lambda}(t) &= \dot{p}_o e^{-t\xi} \left(\frac{\omega^2 - \xi^2}{\omega^3} \sin t\omega + t \frac{\xi}{\omega} \left(\sin t\omega + \frac{\xi}{\omega} \cos t\omega \right) \right), \\ q_{\lambda}(t) &= \dot{p}_o e^{-t\xi} \left(-\frac{1}{\omega^3} \xi^2 \sin t\omega + t \frac{\xi}{\omega} \left(\sin t\omega + \frac{\xi}{\omega} \cos t\omega \right) \right). \end{aligned} \quad (17)$$

where it is evident the critical damped evolution.



Time evolution of $p_{\lambda_{1,1}}(t)$ (impulsive force). Time evolution of $q_{\lambda_{1,1}}(t)$ (impulsive force).

3 Conclusions

In this paper a piezo-electro-mechanical Kirchhoff-Love plate is designed: it is conceived by the introduction of a structural modification based on the concept of transformation of the mechanical vibration energy by means of distributed piezoelectric transducers. In order to obtain a multimode passive vibration controller we synthesize a completely passive electric circuit which is the circuital analog of Kirchhoff-Love plate and employ it to electrically interconnect an array of transducers uniformly distributed on the plate. The improvements with respect to the results found in¹ include: i) the synthesis of a circuit constituted by passive components and ii) a more accurate modelling of the PEM plate to account for those mechanical properties of piezoelectric transducers previously neglected.

The presented results seem to have possible technological applications in the conception of shields for noise control.

We believe that an important aspect of this work can be found in the new methodology adopted in the synthesis of the electric analog by means of a Lagrangian approach, which automatically lead to a completely passive circuit.

Further evolutions of this work are easily found extending the presented methodology in the designing of piezo-electro-mechanic shells.

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