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On the use of a four-dimensional formalism to build hypo-elastic or visco-hypo-elastic constitutive behavior models

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In the present article, we are interested in discussing covariance issues (i.e. frame-indifference) in continuum mechanics, in relation with material objectivity. The latter is presently equivalent to the concept of frame-indifference when considering its classical 3D definition (because it is linked to rigid body motions). In the present paper, it is focused primarily on material constitutive models. The aim is to place the mathematical framework of these constitutive relations within the four-dimensional formalism of general relativity theory. Such an aim has non-negligible interest, for it involves in principle non-trivial issues.

Frame-indifference of material constitutive relations is essential in three-dimensional continuum mechanics for finite transformations (solids or fluids) [1, 2]. Indeed, the presence of a rigid body motion of the matter itself (active transformation) and/or the motion of the chosen frame of reference (passive transformation restricted to Euclidean transformation [1]) may impose constraints on the constitutive relations, if they have to respect this "principle" of material objectivity [3, 4, 5]. Because the two notions (material objectivity and frame-indifference) are similar in three-dimensional continuum mechanics, it has led to strong debates [6, 7] in order to decide whether this "principle" has to be used or not, how and to what kind of materials. A clear explanation and definition of the material objectivity "principle" is provided for example in the classical monograph of Truesdell and Noll [5], while a recent discussion of the encountered problems can be found in [1].

Moreover, differential geometry has been shown to be the more useful mathematical formalism to describe the transformations of continuum media. This has been, in particular, applied to the description of finite transformations of solids [3, 4, 8]. More recently, several authors have made a systematic use of this mathematical formalism within a three-dimensional space (3D) for which Stumpf and Hoppe [9] have proposed a review. Complements can be also found in [10, 5]. This 3D approach has also been systematically applied to the general description of elastic solids [10, 11, 12].

Besides, differential geometry has further found a major application with general relativity for the description of the physical four-dimensional space-time (4D), for applications to gravity interaction [13]. Indeed, the covariance principle guarantees the frame-indifference of all physical relations in any kind of systems of reference (not only the rigid body-like), as developed and reviewed by Landau and Lifshits [14]. It seems thus interesting to write constitutive relations in continuum mechanics using differential geometry, but within this 4D formalism to take advantage of the inherent frame-indifference (i.e. covariance) that it offers. Unlike the 3D approach, material objectivity (invariance to any rigid body superposition) and frame-indifference (invariance to any arbitrary changes of 4D coordinate system) have now to be clearly distinguished within that 4D formalism.

As previously said, it is necessary to ensure the validity of the frame-indifference of behavior models, which can be then properly compared in any frame of reference. The latter is defined in the 3D sense as a rigid body that are only linked each other with rigid translations and rigid rotations. Thus, it is possible to prove that frame-indifference and material objectivity correspond to the same mathematical property [1, 15]. The material objectivity has thus to be used in continuum mechanics. However, it is possible to observe material behaviors depending on the rigid body motion (Barnett effect…). For continuum mechanics of solids such a possibility is in general not observed. Consequently and for instance, for a given kinematics (excluding inertial effect), the Cauchy stress tensor is the same whatever the rotation and/or translation of the material. Thus, the "principal" of material objectivity can be definitively imposed.

Besides, such a "principle" leads to some difficulties. When considering behavior models with rates in 3D, it is necessary to have objective derivatives. This correction of the classical derivative is performed using either the rotation matrix W or the transformation gradient matrix F. It corresponds respectively to the rotational rates or to the convected rates. Infinite of such objective rates can be mathematically constructed, but no real physical arguments can be used to choose one of them. For example, the Jaumann rate and the polar rate (or Green-Naghdi rate) are the most rotational rates used [15]. However, it is impossible to define corresponding derivatives of such rates. It means that behavior models with such a derivative cannot be “properly” integrated. For example, hypo-elastic behavior cannot be integrated in an elastic behavior. In other words, it means that such a hypo-elastic behavior depends on the loading path and the “elastic” behavior may be not reversible. It seriously limits the interest of such rates which are not true derivatives (even for plastic behavior). Eventually, the Jaumann rate presents some specific mechanical response when used under shear loading. The oscillation of this behavior is not realistic.

Nevertheless, application of a rate formalism is of primarily importance for two considerations that are linked to applications for finite transformations. First, it can be used for different materials and behaviors, such as elastomers with hypo-elastic behavior, metals in forming processes with (elasto-)plastic behavior in an incremental tangent approach, biomaterials with viscoelastic response. Second, numerical considerations for tangent approaches need to have more efficient tangent matrix to improve the stability, accuracy and time calculus of simulations. This is possible by using a more efficient derivative formulation.

The covariance principle of differential geometry within a four-dimensional space-time ensures the validity of any equations and physical relations through any changes of frame of reference, due to the definition of the 4D space-time and the use of 4D tensors, operations and operators. This enables to separate covariance (i.e. frame-indifference) and material objectivity (i.e. invariance under superposition of any rigid body motion). Covariant transports are thus defined to serve as tensor rates. These operators can be in particular applied to the Cauchy stress tensor. They have to be used as time derivatives to describe non-linear or dissipative phenomena observed during the finite transformations of a material continuum.

Because an infinite number of rates may be constructed and shown to be objective in 3D, the selection of the appropriate transport and its validity still constitutes an open and debatable question. The present methodology in 4D applied to the formulation of tensor rates allows defining two main derivatives (4D covariant rate and 4D Lie derivative) that can now be justified from a physical point of view. Both, the covariant rate and the Lie derivative are independent of the observer (frame-indifference) and could thus be used in a mechanical constitutive model within a four-dimensional formalism. Therefore, it is possible to build frame-indifferent constitutive models that possibly depend on the superposition of rigid body-motion. We have applied it to the construction of hypo-elastic models. It is possible to construct 4D models equivalent of the existing non-linear 3D models, but also offers the possibility to construct new specific mechanical elastic behavior.

The use of 4D formalism enables to define specific derivatives that can be used either for balance equations (4D covariant derivative) or behavior models (4D Lie derivative), as detailed in [15]. For incremental formulation, the use of Lie derivative is justified from a physical choice (material objectivity), while ensuring the covariance principle. The 4D tensor formalism has been demonstrated to be a correct solution to those problems. In order to prove its interest, different problems of elasticity have been investigated. A method for construction of a 4D hypo-thermo-elastic model has been proposed. Extension to a visco-elastic model has also been performed. Application of this method has been done for a specific material. The use of such a behavior model has been tested for a pure shear solicitation and then compared to results obtained from 3D classical models.

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