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Spatial resolution of wrinkle patterns in thin elastic sheets at finite strain

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Thin elastic sheets develop surface undulations, or wrinkles, in the presence of small compressive strain. The study of this phenomenon has its roots in the pioneering works by Wagner [1] and Reissner [2]. In recent years, interest in thin sheets has greatly increased due to their relevance in a wide array of applications such as biological tissues [3], integrated circuits [4], and solar sails [5, 6, 7]. As a result, wrinkling has recently attracted considerable attention among engineers [8, 9, 10, 11, 12, 13], physicists [14, 15, 16], and biologists [17].

The problem may be regarded as one of characterizing the deformation of a sheet well into the post-buckling range, though it differs from conventional buckling in that it is attended by significant stretching. To model details of the wrinkle patterns such as wavelength and amplitude, an appropriate theory must account for the flexural stiffness of the sheet. This effectively introduces a local length scale in the theory, which in turn figures in the wavelengths of the wrinkles. Typical approaches to characterizing the wrinkled regions in thin sheets in recent years have fallen into three general categories: the tension-field theory [11, 12, 18, 19], analytical approaches based on a Föppl-von Kármán framework [20, 15, 21, 9, 22, 23, 14, 16, 24, 25] and finite element simulations based on bifurcation analyses [10, 13, 26].

We present a well-posed model for the finite bending and stretching of thin sheets derived from, and with optimal accuracy with respect to three-dimensional nonlinear elasticity. We show this model to be tractable for the numerical analysis of general problems by extending the method of dynamic relaxation to thin sheets exhibiting large deformations and wrinkling. We also highlight advantages our model holds over the aforementioned approaches.

Our analysis is based on a model recently derived from three-dimensional nonlinear elasticity which effectively extends Koiter's bending theory [27, 28] to deformations that involve significant stretching [29]. The theory is successfully implemented within a finite difference framework and used to solve a variety of equilibrium problems involving wrinkling. The results show excellent agreement with experimental data available in the literature and demonstrate that the proposed framework is robust and can be effectively used to investigate the behavior of thin sheets.

One such example is inspired by the experiments of Wong and Pellegrino [8]. Here, we simulate the response of a rectangular Kapton sheet clamped along opposite (horizontal) edges, with the remaining (vertical) edges remaining free of load. The considered sheet has dimensions $380mm \times 128mm$, with thickness $h = 0.025mm$ with Young's modulus $E = 3.5GPa$ and Poisson's ratio $\nu = 0.31$ [10]. The lower edge is fixed and the upper edge undergoes a controlled shear (horizontal) displacement. We consider two deformations in which this displacement is either $0.5mm$ or $3.0mm$. The actual and simulated deformed configurations of the sheet are depicted in the first two rows, respectively, of Figure 1. We draw attention to three specific qualitative aspects of the wrinkling seen in the experiments: the wrinkles aligned at nearly 45 degrees in the middle of the sheet, the fan shape transitions at the corners, and the small wrinkles aligned perpendicularly to the unloaded vertical edges. Our simulations capture all of these features. From the cross-sectional plots shown in the third row of Figure 1, we see quantitative agreement as well. In particular, the simulated data on the amplitude and wavelength of the wrinkles furnish remarkable agreement with the experimental data.

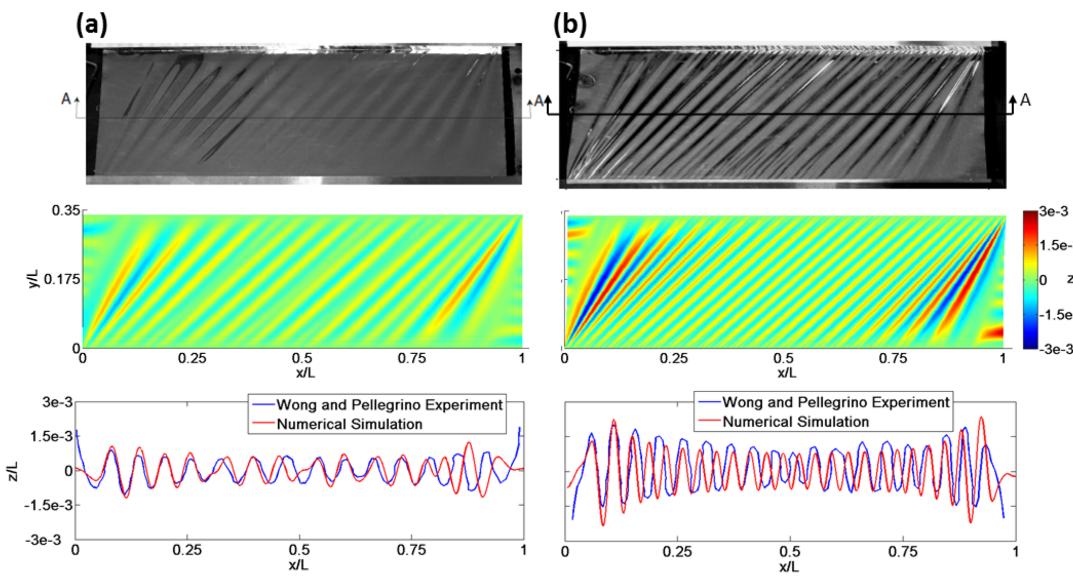

Figure 1. Comparison of simulated sheet deformation with experimental images obtained in [8]. Equilibrium positions shown for a top boundary displacement of 0.05 mm (a) and 3 mm (b). The first row shows the experimental images. The second row shows the corresponding simulated deformation. The third row compares out-of-plane deformation along a cross-section identified as 'A' in the experimental images.

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