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# Interstitial energy flux and stress-power for second-gradient elasticity

**Abstract**
Theories of second gradient elastic materials have been constructed either through the notion of an interstitial energy flux, an additional term to be included in the balance of energy, or through an appropriate extension of the power of internal stresses. Our aim is to propose a critical comparison of such apparently alternative points of view and show how, in some cases, they can be reconciled with each other through an appropriate choice for the expression of the working of the internal stresses.

**Keywords**
Elasticity — Second-gradient — Interstitial energy

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## 1. Introduction

The dependence of constitutive relations for a solid or a fluid on the first and *second* (or higher) deformation gradient, first proposed in some pioneering papers by Toupin [1, 2], poses a well-known conceptual obstacle to the thermodynamical framework of continuum mechanics of so-called non-simple materials, as first shown by Gurtin [3]. The issue has been confronted by means of different approaches, through the introduction either of internal variables [4] or non-standard interaction terms (see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13]).

In particular, motivated by the purpose of describing spatial interaction effects of longer range in elastic materials, Dunn and Serrin developed in a remarkable article [9] a thermodynamic scheme where an additional flux $\mathbf{u}$, the *interstitial energy flux*, is inserted into the balance of energy, beside the heat flow and the standard working of the stress. Such an "extra flux" is not included in the entropy inequality and this framework is then shown to be sufficient for allowing a dependence of constitutive properties on higher-order gradients of deformation. Dunn and Serrin's contribution is all the more interesting in view of the fact that, among other things, a symmetric stress tensor is obtained, thus avoiding the need for micro-polar couples or some other non-standard quantities.

Some Authors have taken the alternative approach of postulating an expression of the stress-power (sometimes called "inner working") which includes one or more additional terms, each one of them linear in the second (or higher) velocity gradient. This idea is basically at the center of the fundamental contribution by Germain [10, 11, 12] and, in some way, is shared by many subsequent developments, due to a variety of Authors (see, e.g., [14, 15, 16, 17]).

This presentation has the much limited aim of drawing the interested reader's attention to a seemingly minor detail which, to our knowledge, might have gone unnoticed in the related Literature (or, at least, among the roughly 140 papers where [9] is cited).

First, we introduce an innocuous slight generalization in the expression of the stress-power: we do not take the "hyperstress" (the tensorial coefficient of the second-gradient of velocity) to be symmetric in the last pair of indexes but allow for a skew-symmetric part which is, of course, powerless, but, interestingly, turns out not be useless. Next, by use of a key technical detail borrowed from the Appendix of [9], we show that this choice makes possible the deduction of constitutive equations for the free-energy $\psi$ and the Piola or Cauchy stresses $\hat{\mathbf{T}}$ and $\hat{\mathbf{S}}$ which are *exactly* coincident with what was derived through the introduction of the interstitial energy by Dunn and Serrin [9]. In particular, the powerless part of the stress-power is what makes the Cauchy stress inherently symmetric, with no need to resort to microcouples.

We believe to be of some interest to know that two different approaches (one based on the interstitial energy and the other one on the introduction of a second-gradient inner power) can be (partially, at least) reconciled. We also think that our computations shed further light on the "inner working" of Dunn and Serrin's approach [9].

We use standard index notation so that our formulas are straightforward an unambiguous (and not so elegant, perhaps), with Greek indexes for the coordinates of points and components of vectors and tensors with respect to a cartesian coordinate system in the reference configuration $\mathscr{B}$, and small Latin indexes for points and components in $\mathscr{B}_t$, the present configuration at current time.

Thus, we shall consistently write $F_{h\alpha}$ and $F_{h\alpha\beta} = F_{h\beta\alpha}$ for the cartesian components of the first and second deformation gradient of a motion described by $x_h(p_\alpha, t)$, with Jacobian $J =$

$\det[F_{h\alpha}]$. As usual, $\rho$ denotes the mass density, with $\rho_0 = \rho J$ the reference density, which we assume to be uniform. The velocity field in the material description is $\dot{x}_h = \partial_t x_h(p_\alpha, t)$ while for the spatial description we write $v_i$ (superposed dots denote material time derivatives, while partial derivatives of a field $\Phi$ with respect to spatial and material coordinates are written as $\Phi_{,h}$ or $\Phi_{,\alpha}$). Thus, $\dot{F}_{h\alpha} = \dot{x}_{h,\alpha}$ and $\dot{F}_{h\alpha\beta} = \dot{x}_{h,\alpha\beta}$. Moreover, $v_{i,j}$ and $v_{i,jk} = v_{i,kj}$ are the first and second spatial velocity gradient.

## 2. The stress-power

In a significant number of contributions, at least since Germain's work [10], the starting point for a discussion of second-gradient materials is an appropriate expression for the stress power (or inner working) associated with a part $\mathscr{P}$. Here, as in [10, 17, 18, 14], we write such quantity as

$$W^{\text{int}}_{\mathscr{P}_t} = \int_{\mathscr{P}_t} [T_{ij} v_{i,j} + G_{ijk} v_{i,jk}] dV, \tag{1}$$

where $T_{ij}$ and $G_{ijk}$ (which is often named hyperstress) are basically just seen as coefficients of $v_{i,j}$ and $v_{i,jk}$. In other words, such tensors are assumed to belong to the dual space of the first and second velocity gradient, in the same way as a force can be seen as a covector whose pairing with the velocity yields the (zeroth-order) working. In particular, quite naturally, it is usually stated, implicitly or explicitly, that, without loss of generality, one may assume symmetry of $G_{ijk}$ with respect to the second and third index

$$G_{ijk} = G_{ikj}. \tag{2}$$

The motivation is almost obvious: a skew-symmetric part (with respect to the same pair of indexes) *would do no work* for any motion of the body and, thus, would appear to be useless and of no effects.

In the case of Germain's article [10] such (entirely reasonable) choice can be deduced when it is stated that $G_{ijk}$ belongs to a space of dimension $3 \times 6 = 18$, while in [17] this is explicitly written in eq. (26) and motivated on p. 521.

The goal of this paper is to find a connection between the modeling of longer range interactions by means of the interstitial work flux, as in [9], and an alternative approach, as in [17, 10, 5], which is based on the stress-power expressed with the introduction of the hyperstress $G_{ijk}$. We show that this second approach leads to an equivalent Cauchy stress tensor and interstitial work flux provided a more general hyperstress is considered, which is not assumed to be symmetric in the sense of (2).

We find more convenient to develop our presentation following a Lagrangian description and, coherently with this choice, we write the stress-power in the reference configuration in the form

$$W^{\text{int}}_{\mathscr{P}} = \int_{\mathscr{P}} [S_{h\alpha} \dot{F}_{h\alpha} + L_{h\alpha\beta} \dot{F}_{h\alpha\beta}]\, dV \tag{3}$$

which, conceptually, is clearly equivalent to (1).

A detailed discussion of the relationship between the alternative expressions for the stress-power provided by (1) and (3) is contained in [18, § 3.3]. As shown there, one can derive both

$$\dot{F}_{h\alpha} = v_{h,k} F_{k\alpha} \tag{4}$$
$$\dot{F}_{h\alpha\beta} = v_{h,k} F_{k\alpha\beta} + v_{h,kl} F_{k\alpha} F_{l\beta} \tag{5}$$

and the inverse relations

$$v_{h,k} = \dot{F}_{h\alpha} F^{-1}_{\alpha k} \tag{6}$$
$$v_{h,kp} = -\dot{F}_{h\alpha} F^{-1}_{\alpha i} F_{i\gamma\beta} F^{-1}_{\gamma k} F^{-1}_{\beta p} + \dot{F}_{h\gamma\beta} F^{-1}_{\gamma k} F^{-1}_{\beta p} \tag{7}$$

In view of (1), (3), (4), (5), (6) and (7), as shown in [18, Prop. 3], condition $W^{\text{int}}_{\mathscr{P}} = W^{\text{int}}_{\mathscr{P}_t}$ for all motions and all parts $\mathscr{P}$ is then guaranteed by relations

$$\begin{aligned} JT_{hk} &= S_{h\alpha} F_{k\alpha} + L_{h\alpha\beta} F_{h\alpha\beta} \\ JG_{hkp} &= L_{h\alpha\beta} F_{k\alpha} F_{p\beta} \end{aligned} \tag{8}$$

which can be inverted as

$$\begin{aligned} S_{h\alpha} &= JT_{hk} F^{-1}_{\alpha k} - L_{h\gamma\beta} F_{k\gamma\beta} F^{-1}_{\alpha k} \\ L_{h\alpha\beta} &= JG_{hkl} F^{-1}_{\alpha k} F^{-1}_{\beta l} \end{aligned} \tag{9}$$

and provide the connection between stress and hyperstress in the Eulerian and Lagrangian description.

As we mentioned before, tensors $G_{ijk}$ and $L_{h\alpha\beta}$, related through (8) and (9), are both usually supposed to be symmetric in last two indexes. The crucial detail of this work is that we are *not* making such an assumption. Indeed, within a Lagrangian description, coherent with (3), we shall find convenient to split the “hyperstress” $L_{h\alpha\beta}$ into a symmetric and a skew-symmetric part (with respect to Greek indexes):

$$L_{h\alpha\beta} = S_{h\alpha\beta} + W_{h\alpha\beta}\,. \tag{10}$$

Of course, while $S_{h\alpha\beta} = S_{h\beta\alpha}$, the skew-symmetric part $W_{h\alpha\beta}$ satisfies the identity

$$W_{h\alpha\beta} = -W_{h\beta\alpha}$$

and, being “powerless” when inserted into (3), seems at first to be useless. It turns out that this is *not* the case, however.

While our contribution does not improve the theory proposed by Dunn & Serrin in [9] we hope it might be of some help in clarifying the relationship between “stress-power” and “interstitial energy”, by pointing out a seemingly unnoticed detail.

## 3. Interstitial energy and stresses

We take as a starting point the (postulated) Lagrangian expression for the stress-power (3) and express such quantity by means of a volume and a surface integral. For a part $\mathscr{P}$ with outward unit normal $m_\alpha$ on the boundary $\partial\mathscr{P}$, through repeated integrations by parts and applications of the divergence theorem we have

$$
\begin{aligned}
W^{\mathrm{int}}_{\mathscr{P}} &= \int_{\mathscr{P}} [S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}]\,dV \\
&= \int_{\mathscr{P}} [(S_{h\alpha}\dot{x}_h)_{,\alpha} - S_{h\alpha,\alpha}\dot{x}_h \\
&\qquad + (L_{h\alpha\beta}\dot{F}_{h\alpha})_{,\beta} - L_{h\alpha\beta,\beta}\dot{F}_{h\alpha}]\,dV \\
&= \int_{\partial\mathscr{P}} [S_{h\alpha}m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,dA \\
&\qquad - \int_{\mathscr{P}} [S_{h\alpha,\alpha}\dot{x}_h + L_{h\alpha\beta,\beta}\dot{F}_{h\alpha}]\,dV \\
&= \int_{\partial\mathscr{P}} [S_{h\alpha}m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,dA \\
&\qquad - \int_{\mathscr{P}} [S_{h\alpha,\alpha}\dot{x}_h + (L_{h\alpha\beta,\beta}\dot{x}_h)_{,\alpha} - L_{h\alpha\beta,\beta\alpha}\dot{x}_h]\,dV \\
&= \int_{\partial\mathscr{P}} [(S_{h\alpha} - L_{h\alpha\beta,\beta})m_\alpha\dot{x}_h + L_{h\alpha\beta}\dot{F}_{h\alpha}m_\beta]\,dA \\
&\qquad - \int_{\mathscr{P}} [S_{h\alpha,\alpha} - L_{h\alpha\beta,\beta\alpha}]\dot{x}_h\,dV.
\end{aligned}
$$

Now, for

$$\hat{S}_{h\alpha} := S_{h\alpha} - L_{h\alpha\beta,\beta} \tag{11}$$

and

$$w_\beta := L_{h\alpha\beta}\dot{F}_{h\alpha}, \tag{12}$$

the stress-power (3) takes the form

$$W^{\mathrm{int}}_{\mathscr{P}} = \int_{\partial\mathscr{P}} \hat{S}_{h\alpha}m_\alpha\dot{x}_h\,dA + \int_{\partial\mathscr{P}} w_\beta m_\beta\,dA - \int_{\mathscr{P}} \hat{S}_{h\alpha,\alpha}\dot{x}_h\,dV$$

or, in absolute notation,

$$W^{\mathrm{int}}_{\mathscr{P}} = \int_{\partial\mathscr{P}} \hat{\mathbf{S}}\mathbf{m}\cdot\dot{\mathbf{x}}\,dA + \int_{\partial\mathscr{P}} \mathbf{w}\cdot\mathbf{m}\,dA - \int_{\mathscr{P}} \mathrm{Div}\,\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}\,dV.$$

In view of the identity

$$\mathrm{Div}(\hat{\mathbf{S}}^t\dot{\mathbf{x}}) = \hat{\mathbf{S}}\cdot\dot{\mathbf{F}} + \mathrm{Div}\,\hat{\mathbf{S}}\cdot\dot{\mathbf{x}}$$

(a superscript $t$ denotes the transpose) the stress-power can be finally written as

$$W^{\mathrm{int}}_{\mathscr{P}} = \int_{\partial\mathscr{P}} \mathbf{w}\cdot\mathbf{m}\,dA + \int_{\mathscr{P}} \hat{\mathbf{S}}\cdot\dot{\mathbf{F}}\,dV = \int_{\mathscr{P}} [\mathrm{Div}\,\mathbf{w} + \hat{\mathbf{S}}\cdot\dot{\mathbf{F}}]\,dV. \tag{13}$$

It is natural to identify $\hat{\mathbf{S}}$, as defined by (11), with the Piola-Kirchhoff stress tensor, in view of its role in the expression (13), and $\mathbf{w}$ with the interstitial energy flux vector (per unit area in the reference configuration) introduced by Dunn and Serrin [9].

In order to better understand such identifications, it is useful to see what would happen had we developed our computations beginning from expression (1). By repeated applications of the divergence theorem to the region $\mathscr{P}_t$ with outward unit normal $\mathbf{n}$ on its boundary $\partial\mathscr{P}_t$, we obtain

$$
\begin{aligned}
W^{\mathrm{int}}_{\mathscr{P}_t} &= \int_{\partial\mathscr{P}_t} \mathbf{u}\cdot\mathbf{n}\,dA + \int_{\mathscr{P}_t} \hat{\mathbf{T}}\cdot\mathrm{grad}\,\mathbf{v}\,dV \\
&= \int_{\mathscr{P}_t} [\mathrm{div}\,\mathbf{u} + \hat{\mathbf{T}}\cdot\mathrm{grad}\,\mathbf{v}]\,dV,
\end{aligned}
$$

where

$$\hat{T}_{hk} := T_{hk} - G_{hkl,l} \tag{14}$$

and

$$u_h := G_{hkl}v_{h,k}. \tag{15}$$

Notice that, from definition (15), in view of (8), (6) and (12), we have

$$
\begin{aligned}
u_l &= J^{-1}L_{h\alpha\beta}F_{k\alpha}F_{l\beta}v_{h,k} = J^{-1}L_{h\alpha\beta}F_{k\alpha}F_{l\beta}\dot{F}_{h\gamma}F^{-1}_{\gamma k} \\
&= J^{-1}L_{h\gamma\beta}F_{l\beta}\dot{F}_{h\gamma} = J^{-1}F_{l\beta}L_{h\gamma\beta}\dot{F}_{h\gamma} \\
&= J^{-1}F_{l\beta}w_\beta,
\end{aligned}
$$

which can be written as $J\mathbf{u} = \mathbf{F}\mathbf{w}$. This relation guarantees that the flux of $\mathbf{u}$ through $\partial P_t$ is the same as the flux of $\mathbf{w}$ through $\partial P$.

It is interesting to notice that $\hat{\mathbf{T}}$, as defined by (14), and $\hat{\mathbf{S}}$, as defined by (11), are connected through the standard relation $J\hat{\mathbf{T}}\mathbf{F}^{-t} = \hat{\mathbf{S}}$, which follows from some rearrangements:

$$
\begin{aligned}
J\hat{T}_{hk}F^{-1}_{\alpha k} &= J(T_{hk} - G_{hkl,l})F^{-1}_{\alpha k} = JT_{hk}F^{-1}_{\alpha k} - JG_{hkl,l}F^{-1}_{\alpha k} \\
&= J[J^{-1}(S_{h\gamma}F_{k\gamma} + L_{h\gamma\beta}F_{k\gamma\beta})]F^{-1}_{\alpha k} \\
&\quad - J[J^{-1}L_{h\gamma\beta}F_{k\gamma}F_{l\beta}]_{,l}F^{-1}_{\alpha k} \\
&= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} - J(J)_{,l}F_{l\beta}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} \\
&\quad - JJ^{-1}(L_{h\gamma\beta}F_{k\gamma}F_{l\beta})_{,l}F^{-1}_{\alpha k} \\
&= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} + J^{-1}J_{,l}F_{l\beta}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} \\
&\quad - L_{h\gamma\beta,l}F_{l\beta}F_{k\gamma}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} \\
&\qquad - L_{h\gamma\beta}F_{k\gamma}F_{l\beta,l}F^{-1}_{\alpha k} \\
&= S_{h\alpha} + L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} + F_{l\rho\beta}F^{-1}_{\rho l}L_{h\gamma\beta}F_{k\gamma}F^{-1}_{\alpha k} \\
&\quad - L_{h\alpha\beta,\beta} - L_{h\gamma\beta}F_{k\gamma\beta}F^{-1}_{\alpha k} - L_{h\gamma\beta}F_{l\beta\rho}F^{-1}_{\rho l}F_{k\gamma}F^{-1}_{\alpha k} \\
&= S_{h\alpha} - L_{h\alpha\beta,\beta} = \hat{S}_{h\alpha}
\end{aligned}
$$

where the identity $J_{,\beta} = JF_{h\alpha\beta}F^{-1}_{\alpha h}$ has been used. We can then regard $\hat{\mathbf{T}}$ as the Cauchy stress tensor provided we prove that $\hat{\mathbf{T}} = \hat{\mathbf{T}}^t$.

The expression for $\mathbf{u}$ given in (15) is a special case of the interstitial energy flux derived in the theory proposed by Dunn and Serrin [9, § 2]. It is important to point out that, as noted by Dell'Isola and Seppecher in the conclusion of [7, §5], the interstitial energy flux $\mathbf{u}$ can be interpreted as the sum of the power of edge contact forces and other types of mechanical interactions. We do not enter into a detailed discussion of this issue, however, and stay within our more limited context.

## 4. Frame indifference of the stress-power

Now, we impose the requirement of frame-invariance on the stress-power per unit volume $W^{\mathrm{int}}$. For $\mathbf{Q} = [Q_{kh}]$ the rotation which connects observers $\mathscr{O}$ and $\mathscr{O}^+$, we easily deduce that

$$F^+_{h\alpha} = Q_{hk}F_{k\alpha} \qquad F^+_{h\alpha\beta} = Q_{hk}F_{k\alpha\beta}$$

from which

$$\dot{F}^{+}_{h\alpha} = \dot{Q}_{hk}F_{k\alpha} + Q_{hk}\dot{F}_{k\alpha}$$
$$\dot{F}^{+}_{h\alpha\beta} = \dot{Q}_{hk}F_{k\alpha\beta} + Q_{hk}\dot{F}_{k\alpha\beta}$$

For

$$W^{\text{int}} = S_{h\alpha}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta}$$
$$W^{\text{int},+} = S^{+}_{h\alpha}\dot{F}^{+}_{h\alpha} + L^{+}_{h\alpha\beta}\dot{F}^{+}_{h\alpha\beta}$$

it follows that

$$W^{\text{int}} = W^{\text{int},+} \Leftrightarrow \begin{cases} S^{+}_{k\alpha} = Q_{kh}S_{h\alpha} \\ L^{+}_{k\alpha\beta} = Q_{kh}L_{h\alpha\beta} \\ S_{h\alpha}F_{k\alpha} + L_{h\alpha\beta}F_{k\alpha\beta} \text{ is symmetric} \end{cases} \tag{16}$$

(The details can be found in [18, Prop. 8, eq. 87]).

As we shall later see, requirement $(16)_3$ is equivalent to frame indifference for the free energy function $\psi(\mathbf{F}, \nabla\mathbf{F}, \theta, \nabla\theta)$. Thus, this is not a condition which we find unnatural.

## 5. Stress-power and balance of angular momentum

We have not yet made any use of (10), which crucially splits $L_{h\alpha\beta}$ into a symmetric and a skew-symmetric part. In this section we show that it is precisely the presence of the powerless skew-symmetric term $W_{h\alpha\beta}$ which makes possible, in general, to obtain a symmetric stress tensor and, thus, balances the angular momentum.

More precisely, as we prove here below, the (seemingly useless) part $W_{h\alpha\beta}$ of $L_{h\alpha\beta}$ is fully determined by condition $\hat{\mathbf{T}} = \hat{\mathbf{T}}^t$ as a function of the symmetric part $S_{h\alpha\beta}$. In a sense, we value this connection to be the main contribution of our article on this topic of second-gradient materials.

The identification of $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ with the Piola and Cauchy stress are completed by proving that

$$\hat{\mathbf{S}}\mathbf{F}^t = \mathbf{F}\hat{\mathbf{S}}^t \tag{17}$$

which amounts to the symmetry of $\hat{\mathbf{T}} = J^{-1}\hat{\mathbf{S}}\mathbf{F}^t$.

In view of (11), condition (17) takes the form

$$S_{h\alpha}F_{k\alpha} - L_{h\alpha\beta,\beta}F_{k\alpha} = S_{k\alpha}F_{h\alpha} - L_{k\alpha\beta,\beta}F_{h\alpha},$$

which can be rewritten as

$$S_{h\alpha}F_{k\alpha} - (L_{h\alpha\beta}F_{k\alpha})_{,\beta} + L_{h\alpha\beta}F_{k\alpha\beta}$$
$$= S_{k\alpha}F_{h\alpha} - (L_{k\alpha\beta}F_{h\alpha})_{,\beta} + L_{k\alpha\beta}F_{h\alpha\beta}.$$

Assuming that $(16)_3$ is satisfied, symmetry of the Cauchy stress tensor is now guaranteed by

$$L_{h\alpha\beta}F_{k\alpha} = L_{k\alpha\beta}F_{h\alpha}. \tag{18}$$

In view of (10), condition (18) can be easily written as

$$S_{h\alpha\beta}F_{k\alpha} + W_{h\alpha\beta}F_{k\alpha} = S_{k\alpha\beta}F_{h\alpha} + W_{k\alpha\beta}F_{h\alpha},$$

or, after multiplication by $F_{i\beta}$, as

$$S_{h\alpha\beta}F_{k\alpha}F_{i\beta} + W_{h\alpha\beta}F_{k\alpha}F_{i\beta} = S_{k\alpha\beta}F_{h\alpha}F_{i\beta} + W_{k\alpha\beta}F_{h\alpha}F_{i\beta}. \tag{19}$$

For

$$D_{hki} := S_{h\alpha\beta}F_{k\alpha}F_{i\beta} \qquad H_{hki} := W_{h\alpha\beta}F_{k\alpha}F_{i\beta}$$

we have

$$D_{hki} = D_{hik} \qquad H_{hki} = -H_{hik}$$

and condition (19) takes the final form

$$D_{hki} + H_{hki} = D_{khi} + H_{khi}$$

or, equivalently,

$$D_{hki} - D_{khi} = H_{khi} - H_{hki}. \tag{20}$$

We now borrow from an idea of Dunn & Serrin [9, Appendix A, p. 122] and permute indexes to obtain

$$D_{hik} - D_{ihk} = H_{ihk} - H_{hik} \tag{21}$$

and

$$D_{kih} - D_{ikh} = H_{ikh} - H_{kih}. \tag{22}$$

A sum of (20), (21) and (22), in view of symmetries and skew-symmetries, gives

$$2D_{hki} - 2D_{ikh} = 2H_{khi}$$

so that

$$W_{k\alpha\beta}F_{h\alpha}F_{i\beta} = S_{h\alpha\beta}F_{k\alpha}F_{i\beta} - S_{i\alpha\beta}F_{k\alpha}F_{h\beta}.$$

It is useful to multiply the above expression by $F^{-1}_{\gamma i}$

$$W_{k\alpha\gamma}F_{h\alpha} = S_{h\alpha\gamma}F_{k\alpha} - F^{-1}_{\gamma i}S_{i\alpha\beta}F_{k\alpha}F_{h\beta} \tag{23}$$

and again by $F^{-1}_{\mu h}$ to obtain

$$W_{k\mu\gamma} = F^{-1}_{\mu h}S_{h\alpha\gamma}F_{k\alpha} - F^{-1}_{\gamma i}S_{i\alpha\mu}F_{k\alpha}. \tag{24}$$

Thus, symmetry of the Cauchy stress tensor is guaranteed by a unique appropriate choice of the skew-symmetric (and “powerless”) part $W_{h\alpha\beta}$ of $L_{h\alpha\beta}$, which can be expressed linearly through the symmetric part $S_{h\alpha\beta}$.

Moreover, we anticipate that the skew-symmetric tensor part $W_{h\alpha\beta}$, as determined by (24), is what makes the flux $\mathbf{w}$ frame indifferent.

## 6. Frame indifference of the interstitial energy flux

Since $\mathbf{w}$ is a material vector field in the reference configuration frame-indifference is satisfied by condition $w^{+}_{\beta} = w_{\beta}$ for all changes of observer. In view of definition (12) and condition (16) we write

$$
\begin{aligned}
w_\beta^+ &= L^+_{k\alpha\beta}\dot{F}^+_{k\alpha} = Q_{kh}L_{h\alpha\beta}(Q_{ks}F_{s\alpha})^{\cdot} \\
&= Q_{kh}L_{h\alpha\beta}\dot{Q}_{ks}F_{s\alpha} + Q_{kh}L_{h\alpha\beta}Q_{ks}\dot{F}_{s\alpha} \\
&= Q^t_{hk}\dot{Q}_{ks}L_{h\alpha\beta}F_{s\alpha} + Q^t_{sk}Q_{kh}L_{h\alpha\beta}\dot{F}_{s\alpha} \\
&= W_{hs}L_{h\alpha\beta}F_{s\alpha} + L_{s\alpha\beta}\dot{F}_{s\alpha} \\
&= L_{h\alpha\beta}F_{s\alpha}W_{sh} + w_\beta
\end{aligned}
$$

(where $W_{hs} = Q_{kh}\dot{Q}_{ks}$ is skew-symmetric and arbitrary).

Thus,

$$
w_\beta^+ = w_\beta \Leftrightarrow L_{h\alpha\beta}F_{s\alpha} = L_{s\alpha\beta}F_{h\alpha}
$$

and this is equivalent to condition (18), discussed before in connection with symmetry of the Cauchy stress $\hat{\mathbf{T}}$. We conclude that frame invariance of $\mathbf{w}$ and symmetry of $\hat{\mathbf{T}}$ are guaranteed by the same property of the stress-power, which is satisfied through the appropriate choice (24) for $W_{h\alpha\beta}$ expressed in (24).

## 7. Balance of energy and entropy inequality

The balance of linear momentum is postulated in the classical form, which locally reduces to

$$
\rho_0\ddot{\mathbf{x}} = \rho_0\mathbf{b}_0 + \operatorname{Div}\hat{\mathbf{S}},
$$

while balance of energy is given by

$$
\frac{d}{dt}\int_{\mathscr{P}}[\rho_0 e + \frac{1}{2}\rho_0\dot{\mathbf{x}}^2]\,dV = \int_{\partial\mathscr{P}}[\hat{\mathbf{S}}\mathbf{m}\cdot\dot{\mathbf{x}} + \mathbf{w}\cdot\mathbf{m} - \mathbf{q}_0\cdot\mathbf{m}]\,dA + \int_{\mathscr{P}}[\rho_0\mathbf{b}_0\cdot\dot{\mathbf{x}} + \rho_0 r]\,dV
$$

where all terms are classical, except for the interstitial energy flux $\mathbf{w}$, which we add following Dunn and Serrin's approach. The local form is obtained as

$$
\rho_0\dot{e} = \rho_0 r + \underbrace{\hat{\mathbf{S}}\cdot\dot{\mathbf{F}} + \operatorname{Div}\mathbf{w}}_{w^{\text{int}}} - \operatorname{Div}\mathbf{q}_0,
$$

where the role of the stress-power is clear.

We follow [9] and write the entropy inequality in the reduced Lagrangian local form as

$$
\rho_0(\dot{\psi} + \eta\dot{\theta}) - \hat{\mathbf{S}}\cdot\dot{\mathbf{F}} - \operatorname{Div}\mathbf{w} + \frac{\mathbf{q}_0\cdot\nabla\theta}{\theta} \le 0. \tag{25}
$$

Of course, the fact that the interstitial energy flux is *not* subjected to the same treatment of the heat flux $\mathbf{q}_0$ is a very touchy point, which is discussed at great length by Dunn and Serrin in [9]. Our goal, here, is much limited and we do not discuss such a delicate issue but only remark that $\operatorname{Div}\mathbf{w}$ is a mechanical power.

## 8. A second-gradient free energy

We make a very simple assumption for the free-energy $\psi$, and take it as a function of the first and second deformation gradient, together with the temperature and its gradient:

$$
\psi(\mathbf{F},\nabla\mathbf{F},\theta,\nabla\theta), \quad \psi(F_{h\alpha},F_{h\alpha\beta},\theta,\theta_{,\gamma}).
$$

Frame-indifference of $\psi$ is expressed by the condition that

$$
\psi(\mathbf{QF},\mathbf{Q}\nabla\mathbf{F},\theta,\nabla\theta) = \psi(\mathbf{F},\nabla\mathbf{F},\theta,\nabla\theta)
$$

for all rotations $\mathbf{Q}$. This is discussed in detail by Dunn & Serrin [9, p. 115] and, here, can be shown to be equivalent to

$$
\psi_{F_{h\alpha}}F_{k\alpha} + \psi_{F_{h\alpha\beta}}F_{k\alpha\beta} = \psi_{F_{k\alpha}}F_{h\alpha} + \psi_{F_{k\alpha\beta}}F_{h\alpha\beta}, \tag{26}
$$

which shall be discussed in more detail in a moment.

It is now useful to compute

$$
\dot{\psi} = \psi_\theta\dot{\theta} + \psi_{F_{h\alpha}}\dot{F}_{h\alpha} + \psi_{\theta_{,\gamma}}\dot{\theta}_{,\gamma} + \psi_{F_{h\alpha\beta}}\dot{F}_{h\alpha\beta}
$$

and

$$
\begin{aligned}
\operatorname{Div}\mathbf{w} = w_{\beta,\beta} = \left(L_{h\alpha\beta}\dot{F}_{h\alpha}\right)_{,\beta} &= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + L_{h\alpha\beta}\dot{F}_{h\alpha\beta} \\
&= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + S_{h\alpha\beta}\dot{F}_{h\alpha\beta} + \underbrace{W_{h\alpha\beta}\dot{F}_{h\alpha\beta}}_{\equiv 0} \\
&= L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} + S_{h\alpha\beta}\dot{F}_{h\alpha\beta}.
\end{aligned}
$$

Thus, the entropy inequality (25) takes the form

$$
\begin{aligned}
&\rho_0\big(\psi_\theta\dot{\theta} + \psi_{F_{h\alpha}}\dot{F}_{h\alpha} + \psi_{\theta_{,\gamma}}\dot{\theta}_{,\gamma} + \psi_{F_{h\alpha\beta}}\dot{F}_{h\alpha\beta} + \eta\dot{\theta}\big) \\
&\quad - \hat{S}_{h\alpha}\dot{F}_{h\alpha} - L_{h\alpha\beta,\beta}\dot{F}_{h\alpha} - S_{h\alpha\beta}\dot{F}_{h\alpha\beta} + \theta_{,\gamma}q^0_\gamma/\theta \le 0
\end{aligned}
$$

and from this, with the usual line of arguments, we deduce

$$
\begin{aligned}
\psi_\theta + \eta = 0, \quad \psi_{\theta_{,\gamma}} = 0, \quad \rho_0\psi_{F_{h\alpha}} &= \hat{S}_{h\alpha} + L_{h\alpha\beta,\beta}, \\
\rho_0\psi_{F_{h\alpha\beta}} &= S_{h\alpha\beta},
\end{aligned} \tag{27}
$$

and the classical condition on the heat flux

$$
\theta_{,\gamma}q^0_\gamma \le 0, \quad \nabla\theta\cdot\mathbf{q}_0 \le 0.
$$

After multiplication by $\rho_0$ and in view of $(27)_{3,4}$, condition (26) is transformed into

$$
\begin{aligned}
&(\hat{S}_{h\alpha} + L_{h\alpha\beta,\beta})F_{k\alpha} + S_{h\alpha\beta}F_{k\alpha\beta} \\
&\qquad = (\hat{S}_{k\alpha} + L_{k\alpha\beta,\beta})F_{h\alpha} + S_{k\alpha\beta}F_{h\alpha\beta}.
\end{aligned}
$$

Finally, definition (11) shows that (26) can be expressed as

$$
S_{h\alpha}F_{k\alpha} + S_{h\alpha\beta}F_{k\alpha\beta} = S_{k\alpha}F_{h\alpha} + S_{k\alpha\beta}F_{h\alpha\beta} \tag{28}
$$

which coincides with $(16)_3$. Thus, frame indifference of the free energy $\psi(\mathbf{F},\nabla\mathbf{F},\theta,\nabla\theta)$ implies the invariance of the stress-power under a change of observer.

We now turn to $(27)_3$ which we use as a starting point for deriving an expression for $\hat{S}_{h\alpha}$. In view of (10), with an integration by parts we have

$$
\begin{aligned}
\hat{S}_{h\alpha} &= \rho_0\psi_{F_{h\alpha}} - L_{h\alpha\beta,\beta} \\
&= \rho_0\psi_{F_{h\alpha}} - S_{h\alpha\beta,\beta} - W_{h\alpha\beta,\beta} \\
&= \rho_0\psi_{F_{h\alpha}} - \rho_0(\psi_{F_{h\alpha\beta}})_{,\beta} - W_{h\alpha\beta,\beta} \\
&= \rho_0\big(\psi_{F_{h\alpha}} - (\psi_{F_{h\alpha\beta}})_{,\beta}\big) - W_{h\alpha\beta,\beta}.
\end{aligned} \tag{29}
$$

From (24) it follows that

$$W_{h\alpha\beta,\beta} = \rho_0\big(F^{-1}_{\alpha k}\psi_{F_{k\gamma\beta}}F_{h\gamma} - F^{-1}_{\beta k}\psi_{F_{k\gamma\alpha}}F_{h\gamma}\big)_{,\beta}$$

and, by substitution in (29), we have

$$\hat{S}_{h\alpha} = \rho_0\big[\psi_{F_{h\alpha}} - (\psi_{F_{h\alpha\beta}})_{,\beta} + \big(F^{-1}_{\beta k}\psi_{F_{k\gamma\alpha}}F_{h\gamma} - F^{-1}_{\alpha k}\psi_{F_{k\gamma\beta}}F_{h\gamma}\big)_{,\beta}\big]$$

which is exactly equal to eq. 3.4 in [9, p. 114].

Finally, if we wish to compute the Cauchy stress tensor $\hat{\mathbf{T}}$ we write

$$\begin{aligned}\hat{T}_{hk} &= \frac{1}{J}\hat{S}_{h\alpha}F_{k\alpha} = \frac{\rho}{\rho_0}\hat{S}_{h\alpha}F_{k\alpha} \\ &= \rho\big(\psi_{F_{h\alpha}}F_{k\alpha} - (\psi_{F_{h\alpha\beta}})_{,\beta}F_{k\alpha}\big) - \frac{\rho}{\rho_0}W_{h\alpha\beta,\beta}F_{k\alpha}\end{aligned} \tag{30}$$

and, from (23), we deduce that

$$W_{h\alpha\beta}F_{k\alpha} = \rho_0\big(\psi_{F_{k\alpha\beta}}F_{h\alpha} - F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big).$$

Moreover, since

$$W_{h\alpha\beta,\beta}F_{k\alpha} = \big(W_{h\alpha\beta}F_{k\alpha}\big)_{,\beta} - \underbrace{W_{h\alpha\beta}F_{k\alpha\beta}}_{\equiv 0} = \big(W_{h\alpha\beta}F_{k\alpha}\big)_{,\beta}$$

from (30) we deduce

$$\hat{T}_{hk} = \rho\big(\psi_{F_{h\alpha}}F_{k\alpha} - (\psi_{F_{h\alpha\beta}})_{,\beta}F_{k\alpha}\big) - \rho\big(\psi_{F_{k\alpha\beta}}F_{h\alpha} - F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big)_{,\beta},$$

an expression which we manipulate into

$$\hat{T}_{hk} = \rho\big[\big(\psi_{F_{h\alpha}}F_{k\alpha} + \psi_{F_{h\alpha\beta}}F_{k\alpha\beta}\big) + \big(F^{-1}_{\beta i}\psi_{F_{i\alpha\gamma}}F_{h\alpha}F_{k\gamma}\big)_{,\beta} - \big(\psi_{F_{h\alpha\beta}}F_{k\alpha} + \psi_{F_{k\alpha\beta}}F_{h\alpha}\big)_{,\beta}\big]. \tag{31}$$

Again, this is exactly the same expression for $\mathbf{T}$ ($\hat{\mathbf{T}}$, here) found in [9, p. 114, eq. $(3.2)_1$]. A glance at (31) confirms that (28) makes $\hat{T}_{hk}$ symmetric.

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