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Power-Conjugation of the Green-Naghdi Rate of the Cauchy Stress and the Deformation Rate: Elasticity Tensor

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Several classes of materials, including industrial elastomers and biological tissues, are commonly modelled as hyperelastic, i.e., the stress is obtained by differentiation of the elastic strain energy potential with respect to the conjugated strain. Due to the complexity arising from material, geometrical and contact-related nonlinearities, a numerical solution of boundary problems by means of the Finite Element Method (FEM) is often imperative. Commercially available FEM software packages offer a limited set of anisotropic potentials and user-defined material subroutines must often be written and coupled with the main code. Within each step of the analysis, a congruent deformation gradient tensor is iteratively updated by the main code and prompted as an input to the subroutine, which returns the appropriate forms of the stress tensor and of the fourth-order elasticity tensor. The updated Lagrangian formulation implemented in ABAQUS/Standard employs the spatial elasticity tensor providing the power-conjugation of the Green-Naghdi rate of the Cauchy stress with the deformation rate [3]. The Cartesian-coordinate representation of this spatial elasticity tensor has been first reported by Simo and Hughes [4], based on a result obtained by Mehrabadi and Nemat-Nasser [2]. Building upon these findings, we elaborated a completely covariant, coordinate-free expression of the same tensor and thoroughly analysed its symmetries.

In modern Continuum Mechanics, a body $\mathcal{B}$ and the space $\mathcal{S}$ are three-dimensional Riemannian manifolds with metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$, respectively (in the trivial case, $\mathcal{B} \subset \mathbb{R}^3$, and $\mathcal{S} \equiv \mathbb{R}^3$), a motion is a smooth map $\chi : \mathcal{B} \times \mathbb{R}_0^+ \to \mathcal{S}$, and no particular reference configuration is considered.

The deformation gradient $\boldsymbol{F}$ has components $F^a{}_A = \chi^a{}_{,A}$ and determinant $J = \det \boldsymbol{F}$. The velocity gradient $\boldsymbol{l} = \operatorname{grad} \boldsymbol{v}$ is valued in $[T\mathcal{S}]^1{}_1$ (i.e., a spatial second-order mixed tensor), and its counterpart $\boldsymbol{l}^\flat = \boldsymbol{g}\,\boldsymbol{l}$ valued in $[T\mathcal{S}]^0_2$ (i.e., fully covariant) can be decomposed into $\boldsymbol{l}^\flat = \boldsymbol{d} + \boldsymbol{w}$, where $\boldsymbol{d}$ is the symmetric strain rate and $\boldsymbol{w}$ is the skew-symmetric spin tensor. The counterparts of these tensors in $[T\mathcal{S}]^1{}_1$ are denoted $\mathfrak{d}$ and $\mathfrak{w}$, respectively. The skew symmetric tensor field $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^{-1}$, where $\boldsymbol{R}$ is the rotation tensor of the polar decomposition $\boldsymbol{F} = \boldsymbol{R}.\boldsymbol{U} = \boldsymbol{V}.\boldsymbol{R}$, is valued in $[T\mathcal{S}]^1{}_1$, and coincides with $\mathfrak{w}$ for rigid motions.

We seek for a tensor field $\mathbb{V}$, valued in $[T\mathcal{S}]^4_0$ (i.e., fully contravariant), such that

$$(\mathfrak{w} - \boldsymbol{\Omega})\boldsymbol{g}^{-1} = \mathbb{V} : \boldsymbol{d}, \tag{1}$$

and, rewriting in covariant formalism the Cartesian-coordinate results by Mehrabadi and Nemat-Nasser [2], employing the special tensor products $\underline{\otimes}$ and $\overline{\otimes}$ defined by Curnier et al. [1], we obtain

$$\begin{aligned}[I_1(\boldsymbol{V})\, I_2(\boldsymbol{V}) - I_3(\boldsymbol{V})]\,\mathbb{V} = \big[& I_1(\boldsymbol{V})^2 (\boldsymbol{V} \,\underline{\otimes}\, \boldsymbol{g}^{-1} - \boldsymbol{g}^{-1} \,\overline{\otimes}\, \boldsymbol{V}) \\ & - I_1(\boldsymbol{V})\,(\boldsymbol{b} \,\underline{\otimes}\, \boldsymbol{g}^{-1} - \boldsymbol{g}^{-1} \,\overline{\otimes}\, \boldsymbol{b}) + (\boldsymbol{b} \,\underline{\otimes}\, \boldsymbol{V} - \boldsymbol{V} \,\overline{\otimes}\, \boldsymbol{b}) \big] : \mathbb{I}^T,\end{aligned} \tag{2}$$

where $\boldsymbol{V}$ is the left stretch, $\boldsymbol{b}$ the left Cauchy-Green deformation, I_k are the invariants, and $\mathbb{I}$ is the fourth-order symmetric identity. As suggested by Eq. (1), tensor $\mathbb{V}$ is skew-symmetric on its first couple of feet (indices), symmetric on its second couple of feet (indices) and lacks diagonal symmetry.

Pushing forward the time derivative of the second Piola-Kirchhoff stress $\boldsymbol{S}$, we obtain the Lie derivative of the Kirchhoff stress $\boldsymbol{\tau}$ (which is related to the Cauchy stress $\boldsymbol{\sigma}$ by $\boldsymbol{\tau} = J\,\boldsymbol{\sigma}$),

$$\mathcal{L}_{\chi}[\boldsymbol{\tau}] = \chi_{*}[(\chi^{*}[\boldsymbol{\tau}])^{\bullet}] = \boldsymbol{F}\,\dot{\boldsymbol{S}}\,\boldsymbol{F}^{T} = \boldsymbol{F}\,[(\boldsymbol{F}^{-1}\,\boldsymbol{\tau}\,\boldsymbol{F}^{-T})^{\bullet}]\,\boldsymbol{F}^{T} = \dot{\boldsymbol{\tau}} - \boldsymbol{l}\,\boldsymbol{\tau} - \boldsymbol{\tau}\,\boldsymbol{l}^{T}. \tag{3}$$

Solving for $\dot{\boldsymbol{\tau}}$, using $\boldsymbol{l} = \mathfrak{d} + \mathfrak{w}$, and subtracting $\mathfrak{w} - \boldsymbol{\Omega}$ from both sides, we obtain the Green-Naghdi rate of the Kirchhoff stress as

$$\boldsymbol{\tau}^{\square} = \dot{\boldsymbol{\tau}} - \boldsymbol{\Omega}\,\boldsymbol{\tau} - \boldsymbol{\tau}\,\boldsymbol{\Omega}^{T} = \mathcal{L}_{\chi}[\boldsymbol{\tau}] + \mathfrak{d}\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\mathfrak{d}^{T} + (\mathfrak{w} - \boldsymbol{\Omega})\,\boldsymbol{\tau} + \boldsymbol{\tau}\,(\mathfrak{w} - \boldsymbol{\Omega})^{T}. \tag{4}$$

Using Eq. (1) and the special tensor products $\underline{\otimes}$ and $\overline{\otimes}$ [1], we obtain, after some manipulation,

$$\boldsymbol{\tau}^{\square} = \mathcal{L}_{\chi}[\boldsymbol{\tau}] + [\boldsymbol{g}^{-1}\,\underline{\otimes}\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\overline{\otimes}\,\boldsymbol{g}^{-1}] : \boldsymbol{d} + [\boldsymbol{i}\,\underline{\otimes}\,(\boldsymbol{\tau}\,\boldsymbol{g}) + (\boldsymbol{\tau}\,\boldsymbol{g})\,\overline{\otimes}\,\boldsymbol{i}] : \mathbb{V} : \boldsymbol{d}. \tag{5}$$

For a hyperelastic material with elastic potential W, the time derivative of the second Piola-Kirchhoff stress $\boldsymbol{S} = (\partial W/\partial \boldsymbol{E})(\boldsymbol{E})$ is related to the time derivative of the Green-Lagrange strain $\boldsymbol{E}$ by the relation

$$\dot{\boldsymbol{S}} = \mathbb{C} : \dot{\boldsymbol{E}} \tag{6}$$

where $\mathbb{C} = (\partial^{2} W/\partial \boldsymbol{E}^{2})(\boldsymbol{E})$ is the material elasticity tensor. The spatial counterpart of this expression is

$$\boldsymbol{\sigma}^{\circ} = J^{-1}\,\mathcal{L}_{\chi}[J\,\boldsymbol{\sigma}] = J^{-1}\,\mathcal{L}_{\chi}[\boldsymbol{\tau}] = \mathbb{c} : \boldsymbol{d}. \tag{7}$$

where $\boldsymbol{\sigma}^{\circ}$ is the Truesdell rate of the Cauchy stress and $\mathbb{c} = J^{-1}\,\chi^{*}[\mathbb{C}]$ is the spatial elasticity tensor. Substituting into Eq. (5), we finally obtain

$$\boldsymbol{\tau}^{\square} = J\,\mathbb{c} : \boldsymbol{d} + [\boldsymbol{g}^{-1}\,\underline{\otimes}\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\overline{\otimes}\,\boldsymbol{g}^{-1}] : \boldsymbol{d} + [\boldsymbol{i}\,\underline{\otimes}\,(\boldsymbol{\tau}\,\boldsymbol{g}) + (\boldsymbol{\tau}\,\boldsymbol{g})\,\overline{\otimes}\,\boldsymbol{i}] : \mathbb{V} : \boldsymbol{d}. \tag{8}$$

which, factorising $\boldsymbol{d}$ on the right, takes the final form

$$\boldsymbol{\tau}^{\square} = \mathbb{B} : \boldsymbol{d} = \left[J\,\mathbb{c} + \boldsymbol{g}^{-1}\,\underline{\otimes}\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\overline{\otimes}\,\boldsymbol{g}^{-1} + [\boldsymbol{i}\,\underline{\otimes}\,(\boldsymbol{\tau}\,\boldsymbol{g}) + (\boldsymbol{\tau}\,\boldsymbol{g})\,\overline{\otimes}\,\boldsymbol{i}] : \mathbb{V}\right] : \boldsymbol{d}. \tag{9}$$

where

$$\mathbb{B} = J\,\mathbb{c} + \boldsymbol{g}^{-1}\,\underline{\otimes}\,\boldsymbol{\tau} + \boldsymbol{\tau}\,\overline{\otimes}\,\boldsymbol{g}^{-1} + [\boldsymbol{i}\,\underline{\otimes}\,(\boldsymbol{\tau}\,\boldsymbol{g}) + (\boldsymbol{\tau}\,\boldsymbol{g})\,\overline{\otimes}\,\boldsymbol{i}] : \mathbb{V} \tag{10}$$

is the spatial fourth-order elasticity tensor associated with the Green-Naghdi rate of the Kirchhoff stress, and has the component form

$$[\mathbb{B}]^{abcd} = J[\mathbb{c}]^{abcd} + g^{ac}\,\tau^{bd} + \tau^{ad}\,g^{bc} + [\delta^{a}{}_{p}\tau^{bh}\,g_{hq} + \tau^{ah}\,g_{hq}\,\delta^{b}{}_{p}]\,[\mathbb{V}]^{pqcd}. \tag{11}$$

As an applicative example to support our findings, a planar biaxial test on a biological tissue specimen was simulated in ABAQUS/Standard, and the results of simulations using a built-in material and a user-defined material subroutine were compared.

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