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Material Counterpart of Darcy's Law in Terms of Differential Forms

Salvatore FEDERICO, Dept Mech & Manufact Eng, The University of Calgary, salvatore.federico@ucalgary.ca
Alfio GRILLO, Dept Mathematical Sciences, Polytechnic of Torino, alfio.grillo@polito.it
Reuven SEGEV, Dept Mechanical Engineering, Ben Gurion University, rsegev@bgu.ac.il

Darcy's law is a spatial equation expressing the first-order theory of fluid filtration within a porous medium, stating that the filtration velocity $\boldsymbol{w}$ is linearly related, through the permeability tensor $\boldsymbol{k}$ (function of fluid viscosity and fluid volumetric fraction), to the force density $\boldsymbol{h}$, given by the sum of the negative of the difference between the pressure gradient $\operatorname{grad} p$ and the body force $\rho_{fT}\,\boldsymbol{f}$ (ρ_{fT} is the true mass density of the fluid, and $\boldsymbol{f}$ is usually gravity) to which the fluid is subjected. In the formulation employed in this study, the filtration velocity (or specific discharge) $\boldsymbol{w}$ is obtained by multiplying the fluid-to-solid relative velocity $\boldsymbol{v}_f - \boldsymbol{v}_s$ by the fluid volumetric fraction ϕ_f, and reads

$$\boldsymbol{w} = \phi_f\,(\boldsymbol{v}_f - \boldsymbol{v}_s) = \boldsymbol{k}\,\boldsymbol{h} = -\boldsymbol{k}\,(\operatorname{grad} p - \rho_{fT}\,\boldsymbol{f}). \tag{1}$$

The pressure gradient $\operatorname{grad} p$ and the body force $\boldsymbol{f}$ are spatial covectors fields, valued in the cotangent space $T_x^\star\mathcal{S}$ at every spatial point $x \in \mathcal{S}$, the permeability $\boldsymbol{k}$ is regarded as a spatial tensor field valued in the space $[T_x\mathcal{S}]_0^2$ of second-order contravariant tensors, the solid velocity $\boldsymbol{v}_s$ and the fluid velocity $\boldsymbol{v}_f$ are spatial vector fields, valued in the tangent space $T_x\mathcal{S}$, and the filtration "velocity" $\boldsymbol{w}$ is regarded as a spatial vector field as well. Notwithstanding this seemingly vectorial structure, the correct material counterpart of Darcy's law depends on the formulation. The formulation of Darcy's law expressed by Eq. (1) is analogical to Ohm's law in Electromagnetism, i.e.,

$$\boldsymbol{j} = \rho\,\boldsymbol{v} = \boldsymbol{\kappa}\,\boldsymbol{e} = -\boldsymbol{\kappa}\operatorname{grad}\Phi, \tag{2}$$

where $\boldsymbol{j}$ is the current density, ρ is the charge density, $\boldsymbol{v}$ is the velocity (vector) field, $\boldsymbol{\kappa}$ is the (contravariant) conductivity tensor, Φ is the scalar (electric) potential, and $\boldsymbol{e}$ is the electric (covector) field.

We emphasise the analogy between these two first-order laws because, in Electromagnetism, it is well-established (see, e.g., [3, 4]) to consider a flux such as $\boldsymbol{j}$ as a two-form, i.e., a spatial field valued in the space $\Lambda_2[T_x\mathcal{S}]$ of skew-symmetric covariant second-order tensors.

This is explained by the fact that *a*) every density-like quantity, such as the charge density ρ in the expression of the current density $\boldsymbol{j}$ or the fluid volumetric fraction ϕ_f in the expression of the filtration velocity $\boldsymbol{w}$, is a three-form, or volume form, i.e., a field valued in the space $\Lambda_3[T_x\mathcal{S}]$ of skew-symmetric covariant third-order tensors, and *b*) the contraction of a three-form with a vector yields a two-form.

In general, (differential) r-forms in an n-dimensional vector space are skew-symmetric r-th order tensors. Because of the skew-symmetry, these objects form a "fusiform" structure [1], in which n-forms (or volume forms) have one independent component and therefore look a lot like scalars, and are indeed often called pseudo-scalars, and $(n-1)$-forms have n independent components and thus look a lot like vectors, and are indeed called pseudo-vectors in the elementary treatment of Physics (think about the angular velocity, the moment of a force, etc). However, n-forms and $(n-1)$-forms do not transform like scalars and vectors, but according to an appropriate Piola transformation. Indeed, volume (n-) forms are the main players in the geometric theory of Riemann integration. In terms of measure theory, they induce measures on their domains and they transform accordingly. Similarly, $(n-1)$-forms play in

$(n-1)$-hyper-surface integration the same role that n forms play in volume integration, and transform according to the what is usually perceived as the "Piola transformation for vector fields", which is in fact the Piola transformation for *pseudo*-vector fields, and is sometimes called Nanson's formula. As mentioned above, an $(n-1)$-form has n components and its restriction to an $(n-1)$-dimensional boundary has a single component. Also, the contraction of an n-form with a vector field yields an $(n-1)$-form.

Let us consider the case where the physical space $\mathcal{S}$ and the body $\mathcal{B}$ are modelled by 3-dimensional manifolds on which no particular Riemannian metric is assumed and, according to the modern approach of Continuum Mechanics, no particular reference configuration is assumed. Let $\chi : \mathcal{B} \to \mathcal{S}$ be a configuration, $\boldsymbol{F}$ with components $F^a{}_A = \chi^a{}_{,A}$ the deformation gradient, and $\mathcal{B}_c = \chi(\mathcal{B})$ be the current placement of the body in $\mathcal{S}$.

Then, the pulled-back three-forms $\chi^*[\phi_f]$ and $\chi^*[\rho]$ may be integrated over the body so that the total apparent volume occupied by the fluid phase in a porous medium and the total charge are given by

$$V_f = \int_{\mathcal{B}_c} \phi_f = \int_{\mathcal{B}} \chi^*[\phi_f] = \int_{\mathcal{B}} J\,\phi_f, \qquad q = \int_{\mathcal{B}_c} \rho = \int_{\mathcal{B}} \chi^*[\rho] = \int_{\mathcal{B}} J\,\rho, \tag{3}$$

where we note that that the single component of $\chi^*[\phi_f]$ is related to that of ϕ_f, and the single component of ρ is related to that of $\chi^*[\rho]$, through the Jacobian $J = \det \boldsymbol{F}$ of χ.

Furthermore, the contraction of the three-forms ϕ_f and ρ with the vector fields $\boldsymbol{v}_f - \boldsymbol{v}_s$ and $\boldsymbol{v}$ yields the two-forms $\boldsymbol{w}$ and $\boldsymbol{j}$, respectively, and their integrals on the boundary $\partial\mathcal{B}_c$ of $\mathcal{B}_c$ are

$$\mathrm{Flow}(\boldsymbol{w}, \partial\mathcal{B}_c) = \int_{\partial\mathcal{B}_c} \boldsymbol{w} = \int_{\partial\mathcal{B}} \chi^*[\boldsymbol{w}], \qquad \mathrm{Flow}(\boldsymbol{j}, \partial\mathcal{B}_c) = \int_{\partial\mathcal{B}_c} \boldsymbol{j} = \int_{\partial\mathcal{B}} \chi^*[\boldsymbol{j}], \tag{4}$$

where it is noted that the components of $\boldsymbol{w}$ and $\chi^*[\boldsymbol{w}]$ and those of $\boldsymbol{j}$ and $\chi^*[\boldsymbol{j}]$ are related through the Piola transform (Nanson's formula). In case a nowhere vanishing n-form is given on $\mathcal{S}$, the $(n-1)$-forms may be represented by (pseudo-) vector fields using contraction and one may write

$$\mathrm{Flow}(\boldsymbol{w}, \partial\mathcal{B}_c) = \int_{\partial\mathcal{B}_c} \boldsymbol{w}\,\boldsymbol{n} = \int_{\partial\mathcal{B}} J\,\boldsymbol{w}\,\boldsymbol{F}^{-T}\boldsymbol{N}, \qquad \mathrm{Flow}(\boldsymbol{j}, \partial\mathcal{B}_c) = \int_{\partial\mathcal{B}_c} \boldsymbol{j}\,\boldsymbol{n} = \int_{\partial\mathcal{B}} J\,\boldsymbol{j}\,\boldsymbol{F}^{-T}\boldsymbol{N}, \tag{5}$$

where $\boldsymbol{w}$ and $\boldsymbol{j}$ are here intended as the pseudo-vectors associated to the corresponding two-forms with the same name, and $\boldsymbol{n}$ and $\boldsymbol{N}$ are the normal covectors to the current boundary $\partial\mathcal{B}_c$ and material boundary $\partial\mathcal{B}$, respectively.

We also note that, in Eqs. (1) and (2), the tensors $\boldsymbol{k}$ and $\boldsymbol{\kappa}$ are actually not in $[T_x\mathcal{S}]^2_0$, but in $(\Lambda_2[T_x\mathcal{S}]) \otimes (T_x\mathcal{S})$, i.e., the first "foot" of the seemingly second-order tensor is actually a two-form, and transforms according to a Piola transformation.

Based on these considerations, the correct material form of Darcy's law, in the classical vectorial formalism, is

$$J\,\boldsymbol{w}\,\boldsymbol{F}^{-T} = J\,\phi_f\,(\boldsymbol{v}_f - \boldsymbol{v}_s)\,\boldsymbol{F}^{-T} = J\,\boldsymbol{F}^{-1}\boldsymbol{k}\,\boldsymbol{h} = [J\,\boldsymbol{F}^{-1}\boldsymbol{k}\,\boldsymbol{F}^{-T}]\,[\boldsymbol{F}^T\,\boldsymbol{h}] \quad \Rightarrow \quad \boldsymbol{W} = \boldsymbol{K}\boldsymbol{H}, \tag{6}$$

where $\boldsymbol{W} = J\,\boldsymbol{w}\,\boldsymbol{F}^{-T}$, $\boldsymbol{W} = J\,\boldsymbol{F}^{-1}\boldsymbol{k}\,\boldsymbol{F}^{-T}$, and $\boldsymbol{H} = \boldsymbol{F}^T\,\boldsymbol{h}$ are the material filtration velocity, permeability, and force density, respectively [2]. The material form of Ohm's law is analogous [5].

The purpose of this work is to move a step toward a unified formalism, which might help formulate models capable of accounting for phenomena characterised by a seemingly different Physics, which nevertheless is described by the very same Mathematics.

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