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A novel procedure for the estimation of critical loads in isotropic incompressible elastic solids

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The framework of the non-linear theory of elasticity is a classical setting for studying bifurcation phenomena in solid bodies; in such a context, materials capable of large elastic deformations are often modeled as incompressible by introducing a kinematic constraint on the set of the admissible deformation gradients.

The usual approach for a bifurcation analysis considers a monotonic loading process ruled by a loading parameter, and assumes that the body remains in a fundamental equilibrium state (primary deformation) until the loading parameter reaches a critical value. At this critical value, bifurcating solutions may arise.

For seeking the critical value of the loading parameter, it is very common to adopt the method of adjacent equilibria, which analyzes – as the load increases – the possibility of other equilibrium solutions in the neighborhood of the primary deformation. To this aim, the properties of the linearized equilibrium equations from the primary deformation have to be studied. In this context, the Hadamard criterion of infinitesimal stability (see [1]) plays the role of indicator of a possible bifurcation from a primary equilibrium. As it is well-known, such criterion establishes that a deformation of an incompressible elastic body is infinitesimally stable (no bifurcation allowed) if the functional

$$\Im = \int_{\Omega} \mathbb{C}\left[\mathrm{grad}\mathbf{u}\right]\cdot \mathrm{grad}\mathbf{u} + \mathrm{p}\ \mathrm{grad}\mathbf{u}^{\mathrm{T}} \cdot \mathrm{grad}\mathbf{u} \qquad (1)$$

is strictly positive for any incremental kinematically admissible displacement field $\mathbf{u}$, where $\mathbb{C}$ is the instantaneous elasticity tensor from the deformed state Ω of the body and p is the pressure related to the incompressibility constraint.

Clearly, during a monotonic loading process, (1) depends on the loading parameter; thus, the first possible bifurcation mode corresponds to the value of the loading parameter (critical value) which renders the Hadamard functional zero for the first time. Notice that the Hadamard criterion has also an important variational counterpart: the non-negativity of the Hadamard functional together with the vanishing of the first variation of the total energy deliver necessary conditions for weak local minimizers of the total energy. Furthermore, the strict positivity of the Hadamard functional together with the fulfillment both of the strong ellipticity condition for $\mathbb{C}$ and of the *complementing condition* for the incremental boundary-value problem yield a sufficient condition for a deformed configuration to be a weak local minimizers (see [2-4]).

The above considerations show that the sign of the Hadamard functional is a crucial issue in bifurcation problems. Unfortunately, the study of the sign of (1) on the whole class of kinematically admissible incremental displacement fields $\mathbf{u}$ is not a simple matter and generally leads to significant analytical difficulties; thus, explicit calculations are possible only in special cases like, for example, when the perturbations are expandable in suitable series, or the boundary value problem benefits from some kind of symmetries.

When the characterization of the sign of the Hadamard functional seems to be prohibitive, it is very helpful to develop a lower bound estimate for the critical load, with the aim of seeking a value of the load below which the Hadamard stability condition is definitely satisfied.

This strategy may be useful in bifurcation problems: one may evaluate the critical load corresponding to a special bifurcation mode by simply studying the sign of the Hadamard functional on the corresponding restricted class of incremental displacements. Then, one may wonder if such a special superposed solution is actually the first bifurcation mode (among other bifurcations) which occurs during the considered loading process. If it is not possible to check the sign of (1) on the whole class of kinematically admissible incremental displacements, a possible answer may be obtained by checking whether the gap between the critical load related to the special bifurcation mode and the lower bound estimate for the critical load is sufficiently small or not.

In the literature, one my find different lower bound estimates for the critical load, developed both referring to compressible or incompressible elastic bodies [5-8]. New studies on this issue are motivated by the need of seeking “better quality” estimates which narrows the “distance” between the load below which bifurcations are not possible (lower bound estimate) and the value of the actual critical load (upper bound estimate) in many significant bifurcation problems.

In [9] we have proposed a new procedure for determine an optimal lower bound estimate for the critical load for compressible elastic solids which improves other proposals in literature.

Here, we construct a novel procedure based on the Korn inequality for determining an optimal lower bound estimate of the critical load in the diffused case of incompressible elastic solids.

As representative applications of our general results, we consider certain boundary-value problems for a homogeneously deformed, incompressible, isotropic and homogeneous elastic body and compare our lower bound estimates to other approaches contained in the literature. Finally, we show how our method leads to the determination of a sufficient condition for the uniform positivity of the Hadamard functional, that is a sufficient condition for a deformed configuration to be a weak local minimizer.

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