The 4th Canadian Conference on Nonlinear Solid Mechanics (CanCNSM 2013)
McGill University, July 23-26, 2013
Montréal, Canada

(Extended abstract) Paper ID 871

The Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient to $\mathrm{SO}(n)$ in the canonical left-invariant Riemannian metric on $\mathrm{GL}(n)$

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1. Introduction

We show that the well-known isotropic Hencky strain energy $\mu\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2$ for the symmetric Biot-stretch $U = \sqrt{F^T F}$ measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$ where $\mathrm{GL}^+(n)$ is viewed as a Riemannian manifold endowed with a left-invariant metric which is also right $\mathrm{O}(n)$-invariant (isotropic), and where the coefficients $\mu, \kappa > 0$ correspond to the shear modulus and the bulk modulus, respectively. Thus we provide yet another characterization of the polar-decomposition $F = RU$, $R \in \mathrm{SO}(n)$, $U \in \mathrm{PSym}(n)$, since the stretch U provides also the geodesic distance to $\mathrm{SO}(n)$ in the euclidean metric, i.e., $\operatorname{dist}^2_{\text{euclid}}(F, \mathrm{SO}(n)) = \|U - 1\!\!1\|^2$, with $\|X\| = \sqrt{\operatorname{tr}(X^T X)}$ denoting the Frobenius matrix norm henceforth. For both the euclidean and the geodesic distance, the orthogonal factor in the polar decomposition is the nearest rotation to F.

2. The geodesic distance to $\mathrm{SO}(n)$

Viewing $\mathrm{GL}(n)$ as a Riemannian manifold endowed with a left invariant metric

$$g_A : T_A\,\mathrm{GL}(n) \times T_A\,\mathrm{GL}(n) \to \mathbb{R} : \; g_A(X,Y) = \left\langle A^{-1}X, A^{-1}Y \right\rangle, \quad A \in \mathrm{GL}(n), \tag{1}$$

for a suitable inner product $\langle\cdot,\cdot\rangle$ on the tangent space $T_{1\!\!1}\,\mathrm{GL}(n) = \mathfrak{gl}(n) = \mathbb{R}^{n\times n}$ at the identity $1\!\!1$, the distance between $F, P \in \mathrm{GL}^+(n)$ can be measured along sufficiently smooth curves. We denote by

$$\mathscr{A} = \{\gamma \in C^0([0,1];\mathrm{GL}^+(n)) \mid \gamma \text{ piecewise differentiable, } \gamma(0) = F, \gamma(1) = P\} \tag{2}$$

the admissible set of curves connecting F and P, and by

$$L(\gamma) = \int_0^1 \sqrt{g_{\gamma(s)}(\dot\gamma(s), \dot\gamma(s))}\,\mathrm{ds} \tag{3}$$

the length of $\gamma \in \mathscr{A}$. Then the geodesic distance $\operatorname{dist}_{\text{geod}}(F,P) = \inf_{\gamma\in\mathscr{A}} L(\gamma)$ defines a metric on $\mathrm{GL}^+(n)$. While it is generally difficult to explicitly compute this distance or to find length minimizing curves, it can be shown [1, 2] that if the Riemannian metric is defined by an inner product of the form

$$\begin{aligned}
\langle X,Y\rangle = \langle X,Y\rangle_{\mu,\mu_c,\kappa} &:= \mu\,\langle \operatorname{dev}\operatorname{sym} X, \operatorname{dev}\operatorname{sym} Y\rangle_{n\times n} + \mu_c\,\langle \operatorname{skew} X, \operatorname{skew} Y\rangle_{n\times n} + \tfrac{\kappa}{2}\operatorname{tr} X \operatorname{tr} Y, \\
\|X\|^2_{\mu,\mu_c,\kappa} &:= \langle X,X\rangle_{\mu,\mu_c,\kappa} = \mu\|\operatorname{dev}\operatorname{sym} X\|^2 + \mu_c\|\operatorname{skew} X\|^2 + \tfrac{\kappa}{2}[\operatorname{tr} X]^2, \\
\mu, \mu_c, \kappa &> 0,
\end{aligned} \tag{4}$$

which is the case if and only if g is right invariant under $\mathrm{O}(n)$ [3], then every geodesic γ connecting F and P is of the form

$$\gamma(t) = F\,\exp(t(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi))\,\exp(t(1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi) \tag{5}$$

for some $\xi \in \mathfrak{gl}(n)$, where $\exp : \mathfrak{gl}(n) \to \mathrm{GL}^+(n)$ denotes the matrix exponential, $\operatorname{sym}\xi = \frac{1}{2}(\xi + \xi^T)$ the symmetric part and $\operatorname{skew}\xi = \frac{1}{2}(\xi - \xi^T)$ the skew symmetric part of ξ. Now, according to the classical Hopf-Rinow theorem, there exists a length minimizing geodesic in $\mathscr{A}$. To obtain such a minimizer γ (and thus the distance $\operatorname{dist}_{\text{geod}}(F,P) = L(\gamma)$), it therefore remains to find $\xi \in \mathfrak{gl}(n)$ with

$$P = \gamma(1) = F\,\exp(\operatorname{sym}\xi - \tfrac{\mu_c}{\mu}\operatorname{skew}\xi)\,\exp((1+\tfrac{\mu_c}{\mu})\operatorname{skew}\xi)\,. \tag{6}$$

Although no closed form solution to (6) is known, the equation can be used to obtain a lower bound[1]

$$\inf_{Q\in\mathrm{SO}(n)} \operatorname{dist}_{\text{geod}}(F,Q)^2 \geq \inf_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} \tag{7}$$

for the distance of $F \in \mathrm{GL}^+(n)$ to $\mathrm{SO}(n)$, as well as an upper bound

$$\operatorname{dist}_{\text{geod}}(F, \operatorname{polar}(F))^2 \leq \|\log(\operatorname{polar}(F)^T F)\|^2_{\mu,\mu_c,\kappa} = \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2\,, \tag{8}$$

where $F = RU$, $R = \operatorname{polar}(F) \in \mathrm{SO}(n)$, $U = \sqrt{F^TF} \in \mathrm{PSym}(n)$ denotes the polar decomposition of F. Finally, we can use a recent optimality result proved by Neff et al. [4]:

Theorem 1. *Let $\|\,.\,\|$ be the Frobenius matrix norm on $\mathfrak{gl}(n)$, $F \in \mathrm{GL}^+(n)$. Then the minimum*

$$\min_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(Q\cdot F)\|^2 = \|\log(\sqrt{F^TF})\|^2 \tag{9}$$

is uniquely attained at $Q = \operatorname{polar}(F)^T$.

A consequence of Theorem 1, combined with (7) and (8), yields the main result [5]:

Theorem 2. *Let g be a left invariant Riemannian metric on $\mathrm{GL}(n)$ that is also right invariant under $\mathrm{O}(n)$, and let $F \in \mathrm{GL}^+(n)$. Then:*

$$\operatorname{dist}_{\text{geod}}(F, \mathrm{SO}(n))^2 = \operatorname{dist}_{\text{geod}}(F, \operatorname{polar}(F))^2 = \mu\|\operatorname{dev}\log(U)\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2\,. \tag{10}$$

Furthermore, for $\mu_c = 0$ (in which case $\operatorname{dist}_{\text{geod}}$ defines a pseudometric on $\mathrm{GL}^+(n)$), Theorem 2 still holds.

References

[1] Alexander Mielke. Finite elastoplasticity, Lie groups and geodesics on SL(d). In Paul Newton, Philip Holmes, and Alan Weinstein, editors, *Geometry, Mechanics, and Dynamics*, pages 61–90. Springer New York, 2002.

[2] P. Neff and R. Martin. Minimal geodesics on GL(n) for left invariant Riemannian metrics which are right invariant under O(n). in preparation, 2013.

[3] R. Bryant. Personal communication, 2013. Mathematical Sciences Research Institute, Berkeley.

[4] P. Neff, Y. Nakatsukasa, and A. Fischle. The unitary polar factor $Q = U_p$ minimizes $\|\operatorname{Log}(Q^*Z)\|^2$ and $\|\operatorname{sym}_*\operatorname{Log}(Q^*Z)\|^2$ in the spectral norm in any dimension and the Frobenius matrix norm in three dimensions. *arXiv:1302.3235*, submitted, 2013.

[5] P. Neff, B. Eidel, and F. Osterbrink. The isotropic Hencky strain energy $\|\log U\|^2$ measures the geodesic distance of the deformation gradient $F \in \mathrm{GL}^+(n)$ to $SO(n)$ in the unique left-invariant Riemannian metric on $\mathrm{GL}^+(n)$ which is also right $\mathrm{O}(n)$-invariant. in preparation, 2013.

[1] We denote by log the principal matrix logarithm, while the expression Log is used to indicate that the infimum is taken over the whole inverse image under exp, i.e. $\inf_{Q\in\mathrm{SO}(n)} \|\operatorname{Log}(QF)\|^2_{\mu,\mu_c,\kappa} = \inf\{\|X\|^2_{\mu,\mu_c,\kappa} : X \in \mathfrak{gl}(n),\ \exp(X) = QF\}$.