1D continua with internal variable as homogenized models for modified Hart's antiparallelograms micro-structures

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A B S T R A C T

In this paper, we show, via an asymptotic expansion, that a 1D continuum model, with an internal variable, is suitable to describe the mechanical behavior of the Inverted Hart's micro-architectures, generalizing the treatment presented in Rizzi et al. (2026). An internal variable is needed to describe the kinematics of the 1D continuum: we find an expression for deformation energy, which depends on the gradient of curvature of a 1D continuum and on the introduced internal variable. We find, as a particular case, the results already presented in Rizzi et al. (2026): when the condensation of kinematical parameters is possible, the energy density of the 1D model depends only on the curvature gradient.

1. Introduction

In the present paper, we study a microstructure whose macro-behavior can be modeled with a 1D continuum, whose kinematical descriptors are the placement and an additional scalar internal variable. Once a process of condensation is performed, and this is possible only for a specific class of external loads, the 1D continuum deformation energy is seen to depend on the gradient of curvature only.

Here we adopt the same terminology used in [1] and denote with MHI the modified Hart's Inversor made up by rigid bars and hinges (see Fig. 1a), MHAM the mechanism obtained by assembling several MHIs as in Fig. 1b, MHAS the structure obtained from a MHAM when its degrees of freedom have been eliminated, and MHAC the 1D continuum associated with a MHAS.

1.1. *Generalized continua as mathematical models for exotic mechanical behavior*

The scholarly debate about generalized continua is a history of controversies (see [2]), tracing back to the pioneering work of Gabrio Piola [3–5] in the 19th century and to the Hellinger's *Fundamentals of the Mechanics of Continua* [6–8]: in fact, too many scholars did wrongly believe that their own approximating assumptions did exhaust completely the modeling capacities of continuum mechanics.

Piola's contributions, largely under-appreciated for more than one century, provided the foundation of the mechanics of continua with internal structure, nonlocal interactions, and higher-gradient effects. After a period of relative neglect, interest in generalized continuum theories reemerged in the early twentieth century, as classical Cauchy continua were beginning to be recognized as insufficient for modeling materials with complex microstructures, multiple-scale interactions and size-dependent behaviors, as witnessed by the work *Fundamentals of the Mechanics of Continua* by Hellinger (see [6–8]).

Among the earliest systematic developments, the fundamental work by Eugène and François Cosserat is very important, as they introduced a theory capable of incorporating independent rotations of material points. Their seminal monograph, *Théorie des corps déformables* (1909), laid the foundation for what would later be called Cosserat or micropolar continua. Cosserats' viewpoint, ahead of their time, remained underappreciated for decades, mainly due to the Cosserats' systematic use of variational principles, which were unfoundedly criticized by several schools of mechanicians (see, e.g., the section about Variational Principles in [9]).

The mid-twentieth century saw a resurgence of interest in enriched kinematics continuum theories: a major contribution came from Raymond D. Mindlin, who developed strain-gradient and microstructure-dependent models that generalized classical elasticity by introducing additional kinematic variables and higher-order gradients. His influential works, including '*Micro-structure in linear elasticity*' (see [10]) and '*Second gradient of strain and surface tension in linear elasticity*' (see [11]), laid the foundations for micromorphic and microstretch theories. Mindlin's approach, like that of the Cosserats, was based on variational principles, emphasizing consistency and mathematical well-posedness in deriving governing equations and the associated boundary conditions.

Building upon these foundations, A. Cemal Eringen unified micropolar and micromorphic theories into a coherent framework, presented comprehensively in his *Microcontinuum Field Theories* (see [12]). Eringen's formalism provided a systematic methodology for treating granular media, composites, and materials exhibiting size effects. However, his approach can be criticized for its reliance on the multiplication of postulates, one balance law per independent kinematic descriptor, which may generate compatibility issues among constitutive relations and complicate the closure of governing equations.

Richard A. Toupin's second gradient formulations [13] preceded and influenced much of the subsequent theory. Unlike Eringen, Toupin employed unifying variational principles that naturally ensured consistency and avoided postulation-related ambiguities. The reader will not be misled by the title of Toupin's work: he was well aware of the fact that second gradients are more general than couple stress continua. However, as the necessary and sufficient equilibrium conditions for couple stress continua can be obtained by postulating simply the balance of forces and couples, while this is not possible for second gradient continua, he did want to keep calling the more general models with the name of the particular ones: this was also aimed to avoid clashes with Trusdellian school, which did not believe that second gradient continua were a well-posed and useful models: they refuse the concept of double forces, which play a crucial role in boundary conditions for second gradient continua.

Similarly, Paul Germain made seminal contributions by extending classical continuum mechanics to account for extra degrees of freedom. By systematically applying the principle of virtual power, which is the basic postulate of Lagrangian mechanics, Germain derived consistent balance laws and boundary conditions for generalized continua with additional kinematical parameters describing micro-structures, such as rotations, couple stresses, or second-order gradients (see [14–16]). Germain's work established a rigorous variational foundation, based on the theory of Schwartz distributions, connecting classical elasticity with the non-Cauchy models first envisioned by Piola and the Cosserats [17,18]. Recent commentaries and translations of Germain's theory have reaffirmed its lasting relevance for modern generalized continua [15,19,20].

In recent decades, generalized continuum mechanics has matured into a versatile and active field. Samuel Forest and collaborators have applied micromorphic and Cosserat models to polycrystalline plasticity and metamaterials, highlighting the necessity of enriched kinematics to capture grain rotations, nonlocal effects, and size-dependent phenomena [21–25].

The development of metamaterials, architected solids, and complex microstructured systems has further motivated the systematic use of generalized continuum models [26–30].

1.2. The synthesis problem of metamaterials

A key (and much-debated, see e.g., [31]) question in this context was whether generalized continuum theories are needed to describe some interesting phenomena. In other words, are there some physical situations where Cauchy continua fail their aim of describing deformation phenomena, so that other models are to be introduced?

Some authors believe to have found some logical inconsistencies in generalized continua theory (these arguments are explicitly resumed in [32]): while these objections, in our opinion, were already confuted by Piola (see [2]) with well-grounded logical arguments, the controversy has been definitively solved by showing that, when dealing with some specific microstructures (see e.g. [26,27,33–38]), the most suitable macroscopic models must be second gradient continua.

Subsequently, the problem of proving that, for every well-posed generalized continuum model (i.e., a model in which a well-behaved deformation energy is postulated, together with compatible external virtual work functionals), it is possible to associate existing microstructures has been posed.

This issue, long debated in the literature, gained clarity when the synthesis problem was formulated: it demands designing those microstructures whose homogenized behavior corresponds to a predetermined generalized continuum model.

First results, in this context, include the synthesis of second-gradient extensional beams [39] and subsequent extensions to pantographic beams, plates, and shells, demonstrating that some higher-gradient continua can indeed be synthesized (see e.g. [40–47]). More recently, in [48], an effort to provide a systematic framework for designing fibrous microstructures that deliver targeted macroscopic responses is presented.

Thus, generalized continuum theories — from Piola and the Cosserat brothers to modern gradient and micromorphic formulations [49] — constitute a mature, experimentally relevant, and conceptually unified approach for leading continuum mechanics to the formulation of models describing the complex behavior of microstructured and architected materials. Their continued development informs the design and synthesis of advanced materials showing exotic mechanical properties.

1.3. A macro-model for generalized Hart's antiparallelograms microstructures

Recently, the problem of synthesis of a 1D continuum whose deformation energy depends on the gradient of its curvature has been addressed (see e.g. [50–52]): we believe that it represents an interesting step towards the synthesis of nth gradient generalized continua.

A possible way to find a micro-structure, whose macroscopic energy depends on the gradient of curvature, can be the following: to find a one-degree-of-freedom linkage mechanism, constituted by bars and rotation links, such that a designated set of its nodes occupies a one-parameter family of circumferences, when the Lagrange parameter is varied.

Once suitable extra constraints are added, the found mechanisms form a structure for which a homogenized 1D continuum can be introduced.

The placement of this continuum will provide approximate positions for the previously designated set of nodes.

Indeed, in this way, when the bars constituting the mechanism are elastically deformable, one gets a deformable structure whose floppy modes (i.e., zero energy configurations) of the homogenized 1D continuum are circumferences: this is exactly what we need as a candidate micro-structure which, after homogenization, can synthesize the searched 1D continuum.

In [1] it was considered a micro-structure, based on the kinematics of Hart's Invertors (the MHIs), for which zero-energy configurations are exactly circumferences (see Figs. 1, 2).

The micro-architecture there considered consists in a one degree of freedom mechanism, the MHAM which, as said before, becomes a structure (the MHAS) when adding suitable extra constraints: in this way the structure's reference configuration becomes one of the configurations allowed to the original mechanism (i.e. either a straight line, when the radius of curvature is infinite, or a circumference, when this radius is finite).

In this paper, we conjecture that a 1D continuum model, with an enhanced kinematics obtained by adding to the placement also an extra internal scalar field (we use the nomenclature taken from [12,14,53]), (the MHAC) is suitable to describe the homogenized behavior of the introduced micro-structure.

We find, via an asymptotic expansion, the homogenized deformation energy density for the conjectured 1D continuum model: we prove that it depends only on the gradient of curvature and on a scalar internal variable, which we associate with the micro-deformation of the 'even' interconnectors of the introduced MHI micro-structure.

To be more precise, the newly introduced internal variable describes the 'even' interconnectors' bending angle. In fact, while in [1] one interconnector every two was rigid and the other was allowed to bend, with

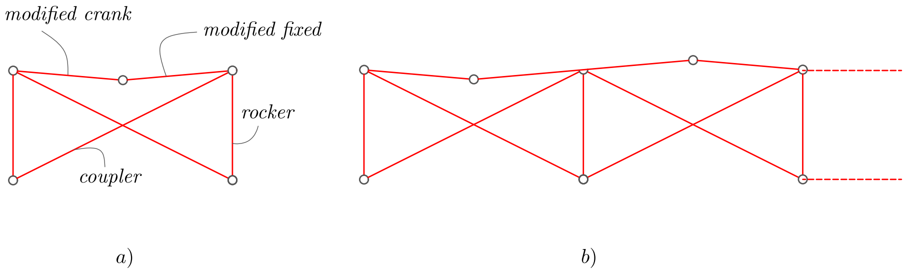

Fig. 1. (*a*) Modified Hart's inversor and (*b*) mechanism scheme.

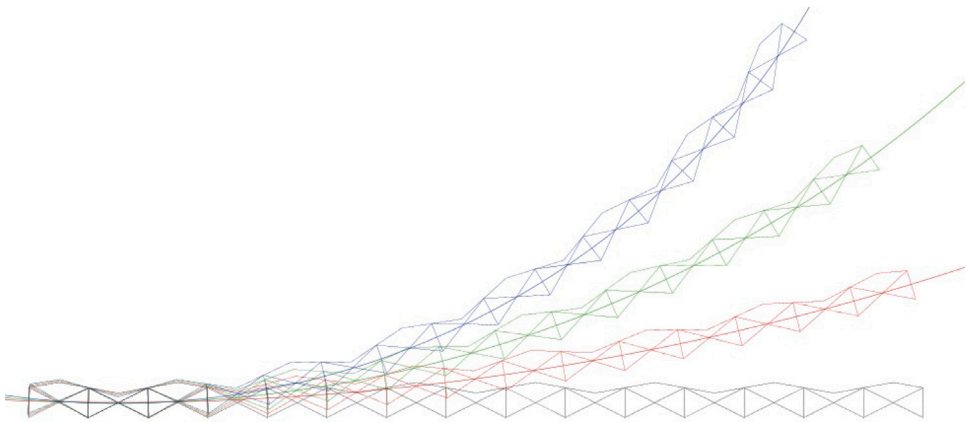

Fig. 2. Kinematics of a MHAM.

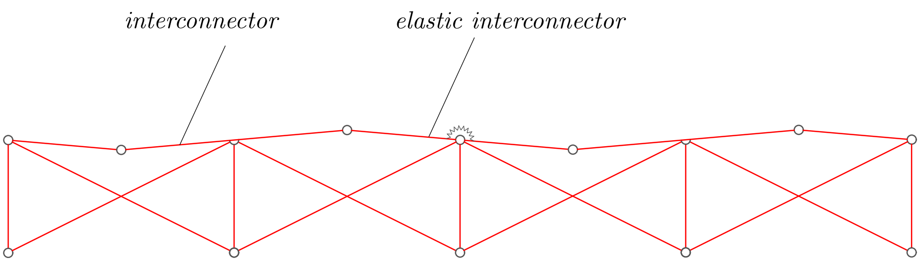

Fig. 3. Two-bars linkage with elastic rotational spring.

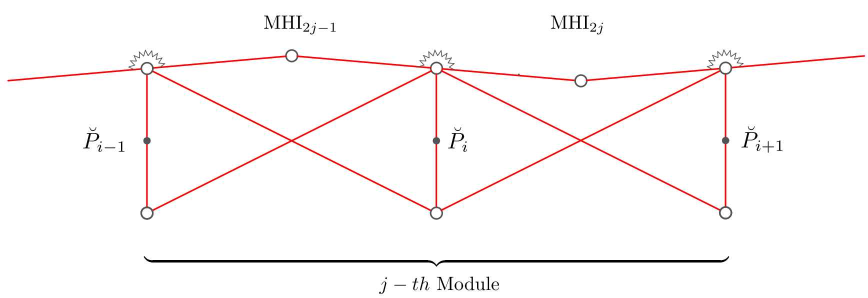

Fig. 4. jth module with rotational springs on all the nodes.

a concentrated rotational stiffness in the node where the interconnector is linked to the rest of the micro-structure (see Fig. 3), in the presently studied micro-structure both even and odd interconnectors can bend (see Fig. 4).

We do not need to introduce a further kinematical descriptor for odd interconnectors because of the validity of a kinematical relationship already published in [1] (see the following Eq. (14)).

We recognize, as a particular case, and once a 'condensation procedure' is performed,[1] that, under the assumption of local equilibrium, the found condensed energy reduces to a third gradient energy, i.e., an

1 This procedure is applicable only when external forces verify the conditions which are specified in the following sections.

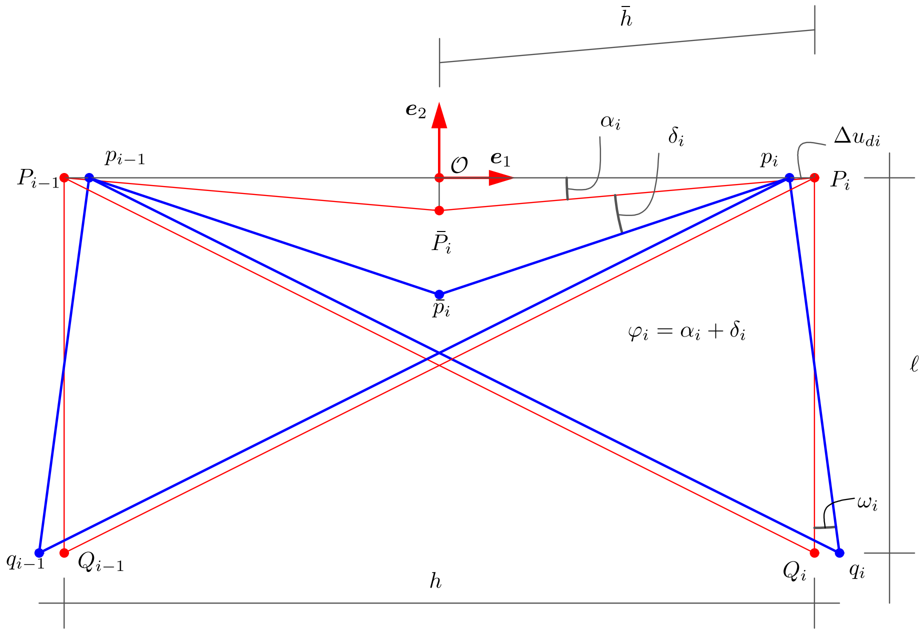

Fig. 5. Reference (in red) and deformed configuration for a given $\delta_i > 0$ (in blue), of a MHI. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

energy which depends on the derivative of the 1D continuum curvature [54,55]. Moreover, the results already presented in [1] are recovered when the stiffness of 'odd' interconnectors tends to infinity.

In this way, we can conclude that also the MHAS, for which both odd and even interconnectors can bend, can be modeled, at macro-scale, as a specific case of third gradient 1D continua: those for which deformation energy depends on the gradient of curvature.

2. Kinematics for considered hart's antiparallelograms micro-structure

In this section, we recall the geometry and the kinematical analysis presented in [1], for seek of self consistency.

The Fig. 5 gives a synthetic description of the MHI which we consider here.

The microstructure is constituted by odd and even antiparallelograms: we assume that in the reference configuration, the even interconnectors have a strictly positive α_i angle (see Fig. 5) while the odd interconnectors have the angle $-\alpha_i$ which is strictly negative (see Fig. 4). Note that the analysis presented in the following cannot be applied to the case in which $\alpha_i = 0$ that deserves a separate treatment.

The odd and even submodules are labeled with an index: odd (even) values of the index refer to odd (even) antiparallelograms (see Fig. 4).

Referring to Fig. 5, we get

$$\begin{aligned} \bar{P}_i &\equiv (0, -\bar{h}\sin\alpha_i) & \bar{p}_i &\equiv (0, -\bar{h}\sin(\varphi_i)) \\ P_i &\equiv (\tfrac{h}{2}, 0) & p_i &\equiv (\bar{h}\cos(\varphi_i), 0) \\ Q_i &\equiv (\tfrac{h}{2}, -\ell) & q_i &\equiv (\tfrac{h}{2} + u_{qi}, -l + v_{qi}) \end{aligned} \tag{1}$$

where

$$\bar{h} = |\bar{P}_i - P_i| = \frac{h}{2\cos\alpha_i} \tag{2}$$

Now, introducing the constraints that represent the inextensibility of the rockers (i.e. the rigid bars connecting the material points P_i and Q_i and P_{i-1} and Q_{i-1}) and the couplers (i.e. the rigid bars connecting the material points P_{i-1} and Q_i and P_i and Q_{i-1})

$$\begin{aligned} (q_i - p_i)^2 &= \ell^2 \\ (q_i - p_{i-1})^2 &= \ell^2 + h^2 \end{aligned} \tag{3}$$

we obtain

$$\begin{aligned} u_{qi} &= \frac{h\left[h\sec\varphi_i - 2\bar{h}\right]}{4\bar{h}} \\ \Delta u_{di} &:= (q_{di} - p_{di})\cdot \boldsymbol{e}_1 = \frac{h^2\sec\varphi_i}{4\bar{h}} - \bar{h}\cos\varphi_i \\ \sin\omega_i &= \frac{h^2\sec\varphi_i}{4\bar{h}\ell} - \frac{\bar{h}}{\ell}\cos\varphi_i = \frac{h}{2\ell}\left(\frac{\cos\alpha_i}{\cos\varphi_i} - \frac{\cos\varphi_i}{\cos\alpha_i}\right) \end{aligned} \tag{4}$$

and finally

$$\omega_i = \arcsin\frac{h}{2\ell}\left(\frac{\cos\alpha_i}{\cos(\delta_i+\alpha_i)} - \frac{\cos(\delta_i+\alpha_i)}{\cos\alpha_i}\right) \tag{5}$$

3. Macro and micro kinematical descriptors

The present instance of MHAM is an articulated linkage having as many degrees of freedom as many MHIs we have. They correspond to the number of the internal interconnectors, plus one i.e. the 'global' degree of freedom of the mechanism, which we have studied in [1], where all bars involved in the linkage, including the interconnectors, are rigid. Of course, there are also the 3 dof that describe the overall rigid motion in 2D, which are inessential in the following analysis.

The bending of odd and even interconnectors being concentrated in their central node, we can use the angles Δ_j (see Fig. 6) as the needed Lagrange parameters to be added to the curvature of the first module of the structure, which is one of the possible 'global' kinematical parameters.

The aim of the present section is to show that, given the macro 'even' curvature of the introduced 1D continuum and an extra macro-scalar kinematical descriptor, describing at the macro-level the bending of even interconnectors, it is possible to univocally determine, at the first order with respect to the size $2h$ of the microstructure module, all micro-structure Lagrange parameters.

3.1. The basic micro-kinematical relationships

The present analysis starts from the following kinematical relationships, which hold for every j (which can be odd or even) and were deduced in the previous section, following the reasoning presented in [1].

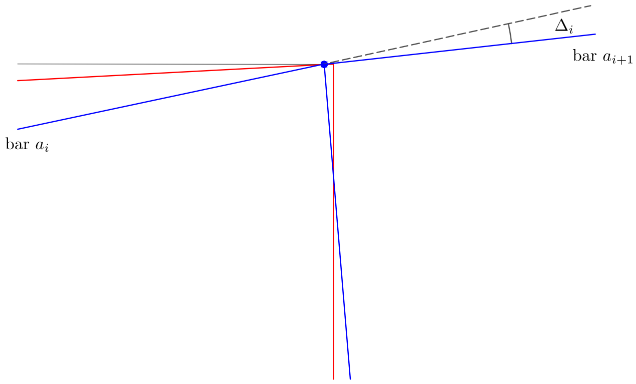

Fig. 6. Interconnector split into two links.

We remark that the functional dependence of the ωs on δs and αs, is independent of j so that we use the notation

$$\omega(\delta_j, \alpha_j) =: \omega_j \tag{6}$$

Now the curvature k_j of the circumference passing through the points $\breve{p}_{i-1}, \breve{p}_i, \breve{p}_{i+1}$ can be easily derived by looking at Fig. 7

$$k_j = \frac{1}{h}(\omega_j + \omega_{j-1}) \tag{7}$$

while the relationship

$$\Delta_j = (1 - r_j)\omega_j + (1 - r_{j-1})\omega_{j-1} \tag{8}$$

has been derived as equation (72) in [1], to which we address the reader.

Note that the scalar quantity r_j is given by the expression

$$r_j = \frac{\ell}{h \tan \alpha_j} \tag{9}$$

(see Fig. 5 hereabove, and eqn. (55) in [1]) and that $\Delta_j = 0$ identify the case in which the interconnector is straight.

As $r_j = r_{j-2}$, the Eqs. (7), (8), lead to

$$\begin{aligned}
\Delta_j - \Delta_{j-1} &= (1 - r_j)\omega_j + (1 - r_{j-1})\omega_{j-1} \\
&\quad - (1 - r_{j-1})\omega_{j-1} - (1 - r_{j-2})\omega_{j-2} \\
&= (1 - r_j)\left(\omega_j - \omega_{j-2}\right) \\
&= (1 - r_j)\left(\omega_j + \omega_{j-1} - \left(\omega_{j-1} + \omega_{j-2}\right)\right) \\
&= h(1 - r_j)\left(k_j - k_{j-1}\right). \qquad (10)
\end{aligned}$$

Now, summing the odd and even instances of the previous equations we obtain

$$\Delta_{2j} - \Delta_{2j-1} = (1 - r_{2j})(k_{2j} - k_{2j-1})h \tag{11}$$

$$\Delta_{2j-1} - \Delta_{2j-2} = (1 - r_{2j-1})(k_{2j-1} - k_{2j-2})h \tag{12}$$

and, taking into account that $r_{2j-1} = -r_{2j}$, we put $r_{2j} = r, r_{2j-1} = -r$, so that we can write,

$$\frac{\Delta_{2j} - \Delta_{2j-2}}{h} = (1 + r)(k_{2j} - k_{2j-2}) - 2r\left(k_{2j} - k_{2j-1}\right) \tag{13}$$

Extracting $k_{2j} - k_{2j-1}$ from this equation and substituting in (11) we get

$$\Delta_{2j-1} = \Delta_{2j} + \frac{1 - r}{2r}\left(h\frac{\Delta_{2j} - \Delta_{2j-2}}{h} - (1 + r)h^2\frac{k_{2j} - k_{2j-2}}{h}\right) \tag{14}$$

Note that this last equation allows for the representation of odd Δs in terms of even Δs and curvatures.

By multiplying the (11) times $(1 + r_{2j})$ and (12) times $(1 - r_{2j})$ and adding the obtained equalities we get

$$(1 + r)\left(\Delta_{2j} - \Delta_{2j-1}\right) + (1 - r)\left(\Delta_{2j-1} - \Delta_{2j-2}\right)$$

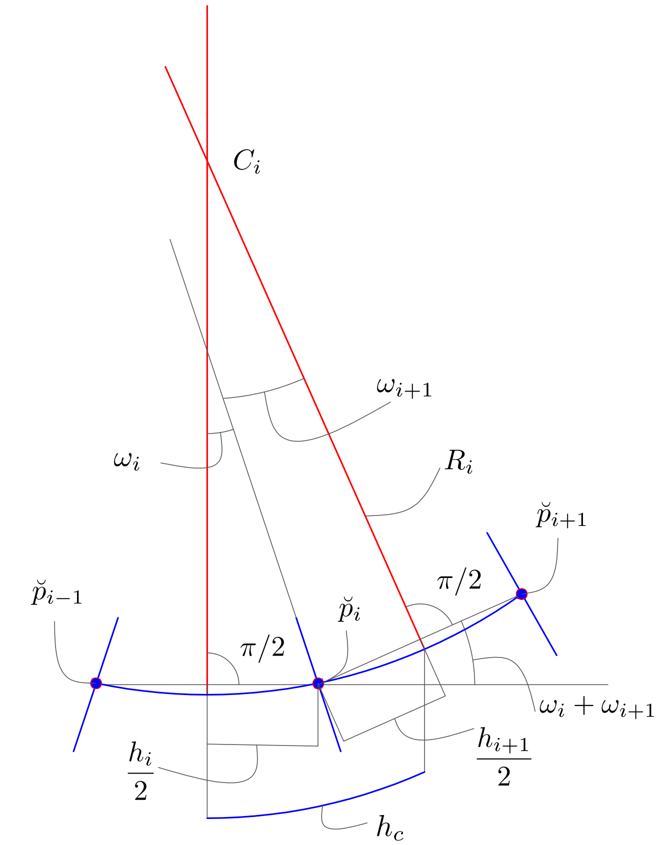

Fig. 7. Centre (C) and radius of curvature (R).

$$= 2h^2(1 - r^2)\frac{(k_{2j} - k_{2j-2})}{2h} \tag{15}$$

which becomes, with simple algebra,

$$\frac{k_{2j} - k_{2j-2}}{2h} = \frac{(1 + r)\Delta_{2j} - 2r\Delta_{2j-1} - (1 - r)\Delta_{2j-2}}{2h^2(1 - r^2)} \tag{16}$$

Moreover we recall the expression of odd ks in terms of even ones, reported as equation (79) in [1], that is

$$k_{2j-1} = \frac{(r - 1)k_{2j} + (r + 1)k_{2j-2}}{2r} \tag{17}$$

3.2. Continualization

Our goal is to study the limit model where the number N of the modules is very large or, in other words, when the size $2h$ of the module is very small compared to the length $L = 2hN$ of the whole structure.

To this end, let us consider the middle points of the rockers in the reference configuration, $\breve{P}_i$ (see Fig. 4), and identify with $(x_j)_{j=0}^{2N}$, $x_o = 0$, their positions. Then, if $x^h \in [0, 2hN]$ is the abscissa of the straight line connecting the points $\breve{P}_i$, we see that

$$x^h = 2hj = x_j \tag{18}$$

In this way, to any function $f(x_j)$, $x_j \in \{0, \dots, 2N\}$, will be associated a function $f^h(x^h)$, $x^h \in [0, 2hN]$, piecewise linear in each interval $[x_j, x_{j+1}]$, so that

$$f^h(2hj) = f(x_j) \tag{19}$$

Then, we will let L fixed and study the limit $h \to 0$ and restrict our analysis to equilibria where the deformation energy remains finite.

4. Deformation energy

Remembering that the MHAS considered here has all rigid bars except interconnectors, we study a simplified structure where the interconnector jth is made by two equal rigid bars interacting via a linear rotational spring of stiffness Y_j. The elastic energy of the microstructure is entirely concentrated in the interconnectors and reads

$$E^h := \frac{1}{2}\sum_{i=1}^{2N-1} Y_i \Delta_i^2 \tag{20}$$

The interaction, in the first and last MHI, of the interconnectors at the boundary of the whole structure will be described at macro level by suitable boundary conditions.

We assume that Y_i are positive.

In the following, it will be clear (see Eq. (26)) that, for the MHAM considered here, it is possible to have configurations with constant curvature having deformation energy different from zero. We will show, however, that under the assumptions made in the condensation process, those can be excluded so that the modes with constant curvature reduce to the global floppy modes, illustrated in Fig. 2 and studied in [1].

We assume, furthermore, that when the material points $\check{P}_0, \check{P}_1$ and $\check{P}_2$ (see Fig. 4) have a fixed position $\check{p}_0, \check{p}_1$ and $\check{p}_2$, the MHAM becomes a MHAS (see Section 3.1).

Depending on the scaling assumed for the stiffnesses Y_i when $h \to 0$, the assumption of finite deformation energy leads to different limit models. If the stiffnesses are too large then the interconnectors behave asymptotically as rigid bars and the limit model is the mechanism studied in [1] which becomes rigid when appropriate boundary conditions are imposed.

If the stiffnesses are too small the structure degenerates. The critical scaling is, as we will see later, when the stiffnesses are of the order h^{-3}. We study two cases:

(i) all the interconnectors have the same stiffness $Y_i = Y h^{-3}$
(ii) odd and even interconnectors have different uniform stiffnesses, respectively $Y_{2i+1} = Y_o h^{-3}$ and $Y_{2i} = Y_e h^{-3}$.

4.1. *The case of uniform bending stiffnesses*

4.1.1. *Asymptotic analysis*

The total deformation energy of the structure is $E^h := \sum_{j=1}^{N} 2h\pi_j$, where

$$\begin{aligned}\pi_j &:= \frac{1}{2} Y h^{-3} \frac{\Delta_{2j-1}^2 + \Delta_{2j}^2}{2h} \\ &= \frac{1}{4} Y h^{-4} (\Delta_{2j-1}^2 + \Delta_{2j}^2)\end{aligned} \tag{21}$$

Using (14) we can write odd Δs in terms of even Δs and curvatures and setting

$$\kappa_{2j} := \frac{1-r}{r}\left((1+r)k_{2j} - h^{-1}\Delta_{2j}\right) \tag{22}$$

we have

$$\pi_j = \frac{Y}{4}\left(\left(\varrho_{2j} - \frac{\kappa_{2j} - \kappa_{2j-2}}{2h}\right)^2 + \left(\varrho_{2j}\right)^2\right) \tag{23}$$

where we have adopted the rescaling

$$\varrho_{2j} := h^{-2}\Delta_{2j} \tag{24}$$

in the same spirit as done for Y_i.

Hence, using the continualisation introduced in Section 3.2, we obtain

$$E^h = \frac{Y}{4}\sum_{j=1}^{N} 2h\left(\left(\varrho^h(2jh) - \left(\kappa^h\right)'(2j+1)h\right)^2 + \varrho^h(2jh)^2\right) \tag{25}$$

leading to the macroscopic limit energy

$$E = \int_0^L \frac{Y}{4}\left(\left(\varrho - \kappa'\right)^2 + \varrho^2\right). \tag{26}$$

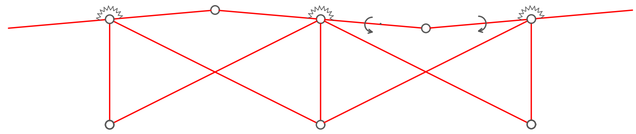

Fig. 8. Couple balance.

Note that in Eq. (25) the sum is extended to the N modules not to the $2N+1$ springs of the structure.

Moreover the identity

$$\kappa_{2j} = \frac{1-r}{r}\left((1+r)k_{2j} - h^{-1}\Delta_{2j}\right) = \frac{1-r}{r}\left((1+r)k_{2j} - h\varrho_{2j}\right) \tag{27}$$

can be written, in terms of interpolating functions, as

$$\kappa^h(2jh) = \frac{1-r}{r}\left((1+r)k^h(2jh) - h\varrho^h(2jh)\right) \tag{28}$$

and passing to the limit for $h \to 0$, we get

$$\kappa = \frac{1-r^2}{r}k \tag{29}$$

Finally, in view of (29) the expression of the deformation energy for the introduced 1D microstructured continuum (26), becomes

$$E = \int_0^L \frac{Y}{4}\left(\left(\varrho - \frac{1-r^2}{r}k'\right)^2 + \varrho^2\right) \tag{30}$$

4.1.2. *Condensation*

If the external loads are applied only on the material central points $\check{P}$ of the vertical links of the MHIs (see Fig. 4), the work of external loads is a linear form of the sequence of the curvatures (see (17)).

In this case, the condensation of macro-deformation energy can be performed, that is the optimization with respect to ϱ. Writing that the strain energy density is stationary with respect to ϱ we get

$$\varrho = \frac{(1-r^2)}{2r}k' \tag{31}$$

and the deformation energy reduces to

$$E = \int_0^L \frac{Y}{2}\left(\frac{1-r^2}{2r}k'\right)^2 \tag{32}$$

One can find an expressive mechanical interpretation of introduced stationarity condition. Indeed, Eq. (16) reads

$$\frac{k_{2n} - k_{2n-2}}{2h} = \frac{\Delta_{2n}(r+1) - \Delta_{2n-2}(1-r) - 2\Delta_{2n-1}r}{2h^2\left(1-r^2\right)} \tag{33}$$

Using the fact that $\Delta_{2n-2} \approx \Delta_{2n}$, Eq. (33) becomes

$$\frac{k_{2n} - k_{2n-2}}{2h} \approx \frac{\left(\Delta_{2n-1} - \Delta_{2n}\right)r}{h^2\left(r^2-1\right)} \tag{34}$$

From (34) we get

$$\Delta_{2n-1} \approx \frac{h^2 k'(2nh)\left(r^2-1\right) + \Delta_{2n}r}{r} \tag{35}$$

Now, assuming that each MHI is self-balanced (see Fig. 8), balance of couples implies

$$\Delta_{2n} = -\Delta_{2n-1} \tag{36}$$

By putting (36) into (35) we obtain

$$\Delta_{2n-1} = \frac{r^2-1}{2r}h^2 k'(2nh) \tag{37}$$

Finally, (35) and (37) give

$$\Delta_{2n} = \frac{1-r^2}{2r}h^2 k'(2nh) \tag{38}$$

that is consistent with (36).

Finally, it is easy to verify that, by putting eqns. (37) and (38) in Eq. (40) one obtains (32).

Therefore, we can conclude that the condensation presented, is possible at least when no external forces and couples are applied to the interconnectors (see again Fig. 8).

4.2. *The case of odd and even different stiffnesses*

We consider now the case (ii) in Section 4.

4.2.1. *Asymptotic analysis*

The total deformation energy of the structure is $E^h := \sum_{j=1}^{N} 2h\pi_j$ where, now

$$\pi_j := \frac{1}{4}h^{-4}(Y_\varnothing \Delta_{2j-1}^2 + Y_e \Delta_{2j}^2). \tag{39}$$

Using (35), Eq. (40) becomes

$$\pi_j := \frac{1}{4}h^{-4}\left(Y_\varnothing\left(\frac{r^2-1}{r}h^2k'(2hn) + \Delta_{2j}\right)^2 + Y_e\Delta_{2j}^2\right) \tag{40}$$

Using the asymptotic assumption (24) we get for the homogenized deformation energy density

$$\pi = \frac{1}{4}\left(Y_\varnothing\left(\varrho - \frac{1-r^2}{r}k'\right)^2 + Y_e\varrho^2\right) \tag{41}$$

and the expression of the deformation energy, which is the counterpart of (30) in the case under consideration, reads

$$E = \int_0^L \frac{1}{4}\left(Y_\varnothing\left(\varrho - \frac{1-r^2}{r}k'\right)^2 + Y_e\varrho^2\right) \tag{42}$$

4.2.2. *Condensation*

Following the assumptions made in Section 4.1.2, we make the derivative of (41) with respect to ϱ, that gives

$$\frac{d\pi}{d\varrho} = \frac{1}{2}\left(\varrho Y_e + Y_\varnothing\left(\varrho - \frac{(1-r^2)\,k'}{r}\right)\right) \tag{43}$$

and, from the stationarity condition, we obtain

$$\frac{d\pi}{d\varrho} = 0 \quad \Rightarrow \quad \varrho = \frac{1-r^2}{r}\frac{Y_\varnothing}{Y_e+Y_\varnothing}k' \tag{44}$$

By substituting (44) in (41) we obtain

$$\pi = \left(\frac{r^2-1}{2r}\right)^2 \frac{Y_eY_\varnothing}{Y_e+Y_\varnothing}k'^2 \tag{45}$$

and, finally, the expression of the deformation energy for the 1D continuum

$$E = \int_0^L \left(\frac{r^2-1}{2r}\right)^2 \frac{Y_eY_\varnothing}{Y_e+Y_\varnothing}k'^2 \tag{46}$$

Also in this case we can find an expressive mechanical interpretation of the stationarity condition.

In order to do that, we can recall the balance of couples that, in this case, must be written as (see Fig. 8)

$$Y_e\Delta_{2n} = -Y_\varnothing\Delta_{2n-1} \tag{47}$$

From (47) we obtain

$$\Delta_{2n} = -\frac{Y_\varnothing}{Y_e}\Delta_{2n-1} \tag{48}$$

and putting (48) into (35), gives

$$\Delta_{2n-1} = \frac{h^2k'(2nh)Y_e\left(r^2-1\right) - Y_\varnothing\Delta_{2n-1}r}{Y_er} \tag{49}$$

and, from Eq. (49)

$$\Delta_{2n-1} = \frac{Y_e}{Y_e+Y_\varnothing}\frac{r^2-1}{r}h^2k'(2nh) \tag{50}$$

Finally, from (35) and (50)

$$\Delta_{2n} = -\frac{Y_\varnothing}{Y_e+Y_\varnothing}\frac{r^2-1}{r}h^2k'(2nh) \tag{51}$$

As the strain energy density, in this case, is

$$\pi_j = \frac{1}{2}h^{-3}\frac{(Y_\varnothing\Delta_{2n-1}^2 + Y_e\Delta_{2n}^2)}{2h} \tag{52}$$

in view of (50) and (51), (52) produces the homogenized energy density

$$\pi = \left(\frac{r^2-1}{2r}\right)^2 \frac{Y_eY_\varnothing}{Y_e+Y_\varnothing}k'^2 \tag{53}$$

that coincides with (45).

Note that, when $Y_e \to \infty$ or $Y_\varnothing \to \infty$ expression (53) reduces to the result that can be obtained using equations (69) and (70) in [1], if $F_j = Y(X_j)$.

In addition, the expression $\frac{Y_eY_\varnothing}{Y_e+Y_\varnothing}$ can be interpreted as the equivalent stiffness of two rotational stiffnesses in series loaded by an assigned couple.

Finally, we want to observe that, given a value of k' in the case analyzed in [1], the sole Δ different from zero was given by expression (56), that is

$$\Delta_{2n-1} = \frac{(r^2-1)}{r}h^2k'(2nh) \tag{54}$$

Comparing (54) with (37) and (38) we observe that, in the case treated in the present paper the same value of k' produces two Δs with the same amplitude (in absolute value), which are one half of (54).

Therefore, as $2(\frac{1}{2})^2 = \frac{1}{2}$, then it is fully justified that the deformation energy density for the microstructure considered in the present paper is $\frac{1}{2}$ of that obtained in [1].

5. Final note

We want to stress that the linearized analysis presented in this paper, is reliable only when the curvature k of each module of the considered MHAS belongs to a suitable interval $(-\bar{k}, \bar{k})$ where $\bar{k} < |k^*|$, k^* being the (negative) value attained by k when $\delta_i = -\alpha_i$ with $\delta_i \in (-\alpha_i, \alpha_i)$. Under this assumption the MHAM, suitably constrained and once introducing elastically bending interconnectors so becoming a MHAS, has a macro-behavior which can be suitably described by a third gradient 1D continuum model.

Indeed, it has to be remarked that the behavior of the MHAM is not the same when its curvature is $k \le k^*$: its study in this last circumstance will be object of future investigations, also by considering the case in which the angle α_i (see Fig. 5) is vanishing. It will be, also, interesting to contrast the results presented in this paper with those obtained for the other micro-mechanism proposed for getting third gradient 1D continua, i.e. ZAPAB mechanism (see [50–52]).

Besides, it is worth noting that equation (63) of [1] still holds, so we can conclude that the MHAC is inextensible also in the case presented here and, consequently, in the macro-deformation energy there is no term related to elongation.

6. Conclusions

In the present paper we show, using a formal asymptotic expansion with respect to the micro-mechanism module size, that, when all the interconnectors are allowed to bend, the most suitable 1D continuum model, apt to describe the macro behavior of the micro-structure, must have an additional scalar kinematical descriptor together with placement: this descriptor takes into account, at macro level, the bending angles of even interconnectors and appears in the homogenized deformation energy linear density. However, using the nomenclature introduced by Eringen (see [12]), we can say that this extra descriptor is a *local internal variable* [56] as its derivative with respect to the

1D continuum curvilinear abscissa does not appear in the deformation energy density. However, we find a heuristic argument, in the framework of the used asymptotic expansion, indicating that, at least for a large class of externally applied loads[2] and in equilibrium situations, the deformation energy density can be condensed reducing to a form in which only the derivative of the curvature appears. Moreover, we establish that, when the odd interconnectors' stiffnesses tend to infinity, we recover the form of deformation energy density established in [1] and a suggestive consideration can be made: exactly as the nonlinear Timoshenko model reduces to the Euler Elastica model if the shear stiffness tends to infinity, if the odd (or equivalently even) interconnectors' stiffnesses tend to infinity, then the 1D continuum with *bending* internal variable reduces to a 1D continuum whose deformation energy depends on the curvature gradient only.

The results presented in this paper will motivate further investigations in several related fields:

1. it will supply an interesting example of a successful solution of a synthesis problem, so giving a benchmark for future research aiming to find algorithms for synthesizing further generalized continua (see e.g. [57–63])
2. it will motivate the development of 3D printing techniques aiming to produce specimens with which to validate experimentally the possibility of exploiting the concept conceived here in technological applications, also using the sensitivity stochastic mechanics methods (see e.g. [64–71])
3. it will demand the application of rigorous mathematical convergence methods (see e.g. [72–74]) to prove that the heuristic asymptotic expansion used here does produce numerically useful reduced continuum models (see e.g. [75–84])
4. it opens the possibility of studying stability results for a novel realistic 1D continuum (e.g., using the techniques presented in [85–89])
5. it justifies the study of wave propagation and solitary waves in third gradient continua, especially when the wavelength of considered vibrations is greater than their characteristic lengths [90–94].

CRediT authorship contribution statement

Francesco dell'Isola: Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Pierre Seppecher:** Writing – review & editing, Supervision, Methodology, Investigation, Conceptualization. **Nicola L. Rizzi:** Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

[2] Those which do not interfere with the microbalance of couples for the parts of the interconnector inside a MHI of the micro-structure module (see Fig. 8).

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