Roberto Fedele[1]
Professor
Department of Civil and Environmental Engineering (DICA),
Politecnico di Milano,
Piazza Leonardo da Vinci 32,
20133 Milan, Italy
e-mail: roberto.fedele@polimi.it

Francesco dell'Isola
Professor
Department DICEAA,
Università dell'Aquila,
Piazzale Ernesto Pontieri, Monteluco di Roio,
67100 L'Aquila, Italy;
M&MoCS International Research Center,
Università dell'Aquila,
67100 L'Aquila, Italy
e-mail: francesco.dellisola@univaq.it

Emilio Barchiesi
Professor
Department of Architecture, Design and Urban Planning,
Università di Sassari,
Piazza Università 21,
07100 Sassari, Italy
e-mail: ebarchiesi@uniss.it

A Historical Perspective on Betti's Theorem and Its Connection With the Potential Theory

In this article, a central result in the theory of linearized elasticity, namely, the reciprocity theorem, is outlined for a continuous solid body, performing the exegesis of the original formulation by Enrico Betti (1823–1892), comparing the proofs he provided with those permitted by modern approaches in continuum mechanics. For linear elastic structures under small displacements, rotations, and strains, legitimating the superposition of the structural responses under multiple loading systems, in the absence of volume forces Betti's theorem allows one to deduce relations involving only loading and displacements over the boundary surface, without requiring the knowledge of the body's elastic stiffness nor making reference explicitly to the distribution of strains and stresses (as it occurs for the conventional formulation of the virtual work). We discuss the auxiliary use of this theorem made by Betti in his works on continuum mechanics, inspired by the potential theory, in particular involving Green's identities. The outlined theoretical strategies, resting on the rigorous deduction of analytical expressions, can nowadays represent a reference for scholars prevalently devoted to repetitive numerical simulations, and an indispensable tool to investigate truly innovative material models. [DOI: 10.1115/1.4071778]

Keywords: constitutive modeling of materials, elasticity, structural mechanics

1 Introduction

Enrico Betti (born in 1823, in Pistoia; died in 1892, in Soiana, close to Pisa, see Fig. 1) was an eminent Italian mathematician, who held high responsibility roles in several institutions, at an academic, educational, and political level, and contributed significantly to the formation of a transnational scientific community [5]. He gave his lectures on various branches of mathematical sciences at the University of Pisa, without interruption from 1857 to 1891 [4,6]. The Opera Omnia of Enrico Betti was edited by the Academia dei Lincei a few years after his death, and published in two volumes, see Refs. [1,7]. All his works (a few frontispieces are shown in Fig. 2), ranging from algebra, through analysis to mathematical physics, reveal an extraordinary capability of either attaining new results or specifying novel methods. Betti oriented his research interests also under the influence of Bernhard Riemann (1826–1866), to whom he was bound by a sincere friendship, see Ref. [8]. The role of physics in his mathematical work became so important that Vito Volterra remarked: "Ceux qui ont connu Betti, non seulement par ses travaux, mais aussi par sa conversation, savent que s'il parlait Mathématiques, bien souvent il pensait Physique" [9].

The potential theory represented, at that time, a topic of prominent importance, strictly related to the theory of partial differential equations. A harsh debate involved the scientific community about the validity of Dirichlet's principle, stating that the solution of the Laplace (or Poisson) equation under Dirichlet boundary conditions (i.e., prescribing the value of the unknown function on the domain boundary) can always be provided by minimizing a suitable energy functional, see, e.g., Ref. [10]. Actually, the functional analysis was not yet sufficiently developed, and it was necessary to attend the beginning of the new century to find a sufficient condition, provided by David Hilbert through a direct method. In his writings on potential theory, mainly published in 1863–1864, Betti adopted a prudent attitude: whenever he could, he privileged the use of Green's function for questions concerning potential functions. In this respect, Betti and Beltrami were confident that Green's theorem could have played a key role in potential theory, and both tried to generalize it. Beltrami extended it to curved manifolds, while Betti used the Green's function to face advanced problems in heat diffusion, elasticity, and sound propagation, see, e.g., Ref. [11].

2 Reciprocity Theorem

Betti's theorem, also referred to as "reciprocity theorem," is known as one of the theoretical results most fecund of applications in the modern mathematical theory of elasticity. It was developed in the second part of his career, during which his research was oriented toward the theory of potential and the problems of mathematical physics in general, see also Ref. [12]. In the contribution

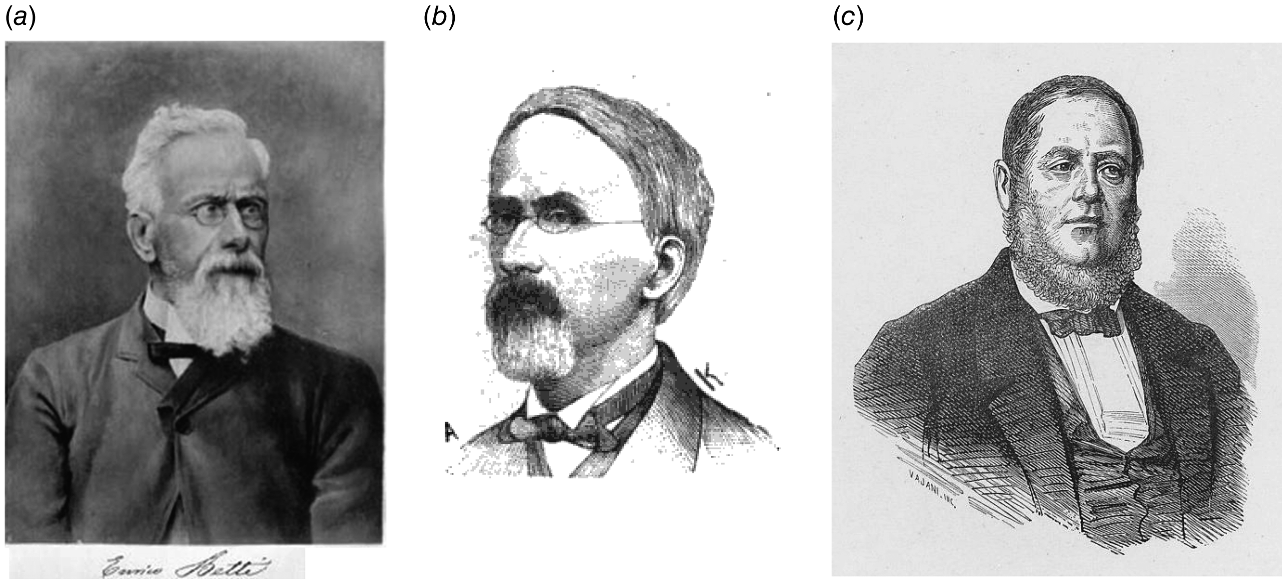

Fig. 1 Photographs and portraits of Enrico Betti: (*a*) extracted from Ref. [1], while (*b*) and (c) are extracted from the books [2,3]. For further documents, see also Ref. [4].

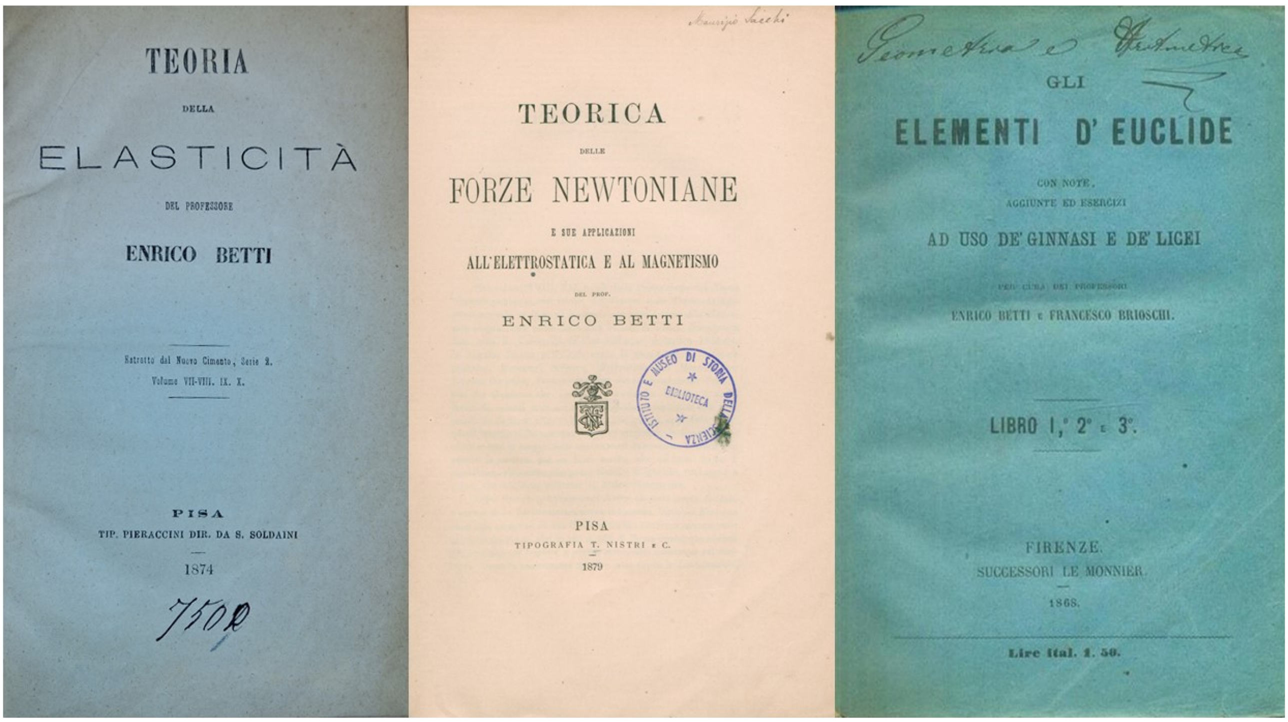

Fig. 2 Frontispiece of some publications of Enrico Betti

published in 1872, titled “Theory of elasticity” (*Teoria dell’elasticità*), see Ref. [7], he stated a relation between two sets of external loading acting on the surface of the same structure under the assumption of linearized elasticity (i.e., with displacements, rotations, and strains assumed to be small). The statement is outlined in Section VI of this publication, titled “General theorem about the deformations equilibrating forces acting exclusively on the surfaces” (*Teorema generale intorno alle deformazioni che fanno equilibrio a forze che agiscono soltanto alle superficie*). Here is the formulation translated into English: “If within a homogeneous, elastic solid body, two systems of displacements specify the equilibrium configuration under two systems of forces acting on the surface, the sum of the products of the force components belonging to the first system by the displacement components of the same points due to the second system is equal to the sum of the products of the components of the forces of the second system by the displacement components of the same points due to the first system.”[2]

[2]“Se in un corpo solido elastico omogeneo, due sistemi di spostamenti fanno equilibrio a due sistemi di forze applicate alla superficie, la somma dei prodotti delle componenti delle forze del primo sistema per le componenti degli spostamenti negli stessi punti del secondo è uguale alla somma dei prodotti delle componenti delle forze del secondo sistema per le componenti degli spostamenti nei medesimi punti del primo” ([7], page 321 et seq.).

The formula written by Betti (using his own notation) is

$$\int_{\sigma}\left(L'v''+M'u''+N'w''\right)d\sigma=\int_{\sigma}\left(L''v'+M''u'+N''w'\right)d\sigma \quad (1)$$

where σ denotes the outer boundary of the body, L', M', N' indicate the components of the surface density of the external actions for the first system, and u', v', w' indicate the corresponding displacement components, while L'', M'', N'' and u'', v'', w'' indicate the relevant entities of the second system. It is worth emphasizing that the vector and matrix formulation at that time was not yet fully developed, and this poses some difficulty for the modern reader: the density of the external virtual work can be more easily expressed through the scalar product as $\mathbf{f}\cdot\delta\mathbf{s}=f_i\,\delta s_i$.[3]

Let us outline the proof by Betti, following the original notation and reasoning path. Let us consider a linear elastic and homogeneous body occupying in the (reference) configuration the volume $S\subset\mathcal{R}^3$, with outer boundary denoted by the symbol σ ($\equiv\partial S$). We indicate by u', v', w' and u'', v'', w'' the components of the (virtual) displacement fields that balance the surface force densities with components L', M', N' and L'', M'', N'', respectively, applied separately on σ. Body forces are absent in both the systems, namely, by Betti's notation, one has $X'=Y'=Z'=X''=Y''=Z''=0$. Although Betti knew very well the virtual work principle by Lagrange,[4] not always he utilized the symbols $\delta u'$, $\delta v'$, $\delta w'$ to denote the virtual displacements (e.g., see Ref. [7], Section III, page 306 et seq.), and mostly he assumes it implicitly. For the structure subjected to the first system of loading, the strain components (specified by the apex $'$) are defined as follows:

$$\begin{gathered}\frac{\partial u'}{\partial x}=a',\quad \frac{\partial v}{\partial y}=b',\quad \frac{\partial w'}{\partial z}=c',\quad \frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}=2f',\\ \frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}=2g',\quad \frac{\partial u'}{\partial y}+\frac{\partial v'}{\partial x}=2h'\end{gathered} \quad (2)$$

The potential energy density for a homogeneous body is a quadratic form of the above strain components: the general expression used by Betti is $P'=\sum_r\sum_s A_{rs}a_r'a_s'$ (in Ref. [7], page 323, Eq. (6) et seq.). When the linearly elastic body turns out to be isotropic, the potential energy depends exclusively on the strain invariants of the first and second orders, say $\Theta=\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=a'+b'+c'$ (volumetric strain) and $\Lambda=\sum_r a_r'^2$, expressed by Betti as

$$\begin{aligned}P'=P'(\Theta,\Lambda)=&-\lambda(a'+b'+c')^2-\mu(a'^2+b'^2+c'^2+2f'^2\\&+2g'^2+2h'^2)\le 0\end{aligned} \quad (3)$$

being μ and λ the Lamé moduli.[5] In modern index notation, we could write $\boldsymbol{\varepsilon}=\mathrm{sym}(\mathrm{grad}\,\mathbf{s})$, being $\mathbf{s}=\{s_i\}$ ($i=1,2,3$) the displacement vector, $\Theta=\boldsymbol{\varepsilon}:1=\varepsilon_{kk}=\mathrm{div}(\mathbf{s})=s_{k,k}$, $\Lambda=\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}=\varepsilon_{lm}\varepsilon_{lm}$. In Eq. (3), Betti considers the potential energy nonpositive: this assumption is consistent with the Lagrangian tradition that seeks the maximum of the total potential energy to identify the equilibrium configuration. The stress components can be deduced by differentiating the energy potential P' with respect to the strain components.

For the first loading condition, the equations stating the local equilibrium under translation (balance of forces) for the volume element in terms of stress components can be expressed as

$$\begin{aligned}&\frac{\partial}{\partial x}\frac{\partial P'}{\partial a'}+\frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial h'}+\frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial g'}=0\\&\frac{\partial}{\partial x}\frac{1}{2}\frac{\partial P'}{\partial h'}+\frac{\partial}{\partial y}\frac{\partial P'}{\partial b'}+\frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial f'}=0\\&\frac{\partial}{\partial x}\frac{1}{2}\frac{\partial P'}{\partial g'}+\frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial f'}+\frac{\partial}{\partial z}\frac{\partial P'}{\partial c'}=0\end{aligned} \quad (4)$$

while over the outer surface, having as normal vector components (pointing inward) α, β, γ, one has

$$\begin{aligned}L'&=\frac{\partial P'}{\partial a'}\alpha+\frac{1}{2}\frac{\partial P'}{\partial h'}\beta+\frac{1}{2}\frac{\partial P'}{\partial g'}\gamma\\M'&=\frac{1}{2}\frac{\partial P'}{\partial h'}\alpha+\frac{\partial P'}{\partial b'}\beta+\frac{1}{2}\frac{\partial P'}{\partial f'}\gamma\\N'&=\frac{1}{2}\frac{\partial P'}{\partial g'}\alpha+\frac{1}{2}\frac{\partial P'}{\partial f'}\beta+\frac{\partial P'}{\partial c'}\gamma\end{aligned} \quad (5)$$

More easily, using the index notation with the Einstein convention, we can write for the energy density $\omega=1/2\,\sigma_{ij}\varepsilon_{ij}=1/2\,D_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$ (being $i,j,k,l=1,2,3$), and hence $\sigma_{ij}=\partial\omega/\partial\varepsilon_{ij}$, $\sigma_{ij,j}=0$ and $\sigma_{ij}n_j=f_i$, being $\mathbf{n}=\{-\alpha,-\beta,-\gamma\}$ the outward normal and $\mathbf{f}=\{L,M,N\}$ the surface force density.

Let us multiply Eq. (4) by the homologous displacement components generated in the same structure by the second loading system, say u'', v'', w'', and integrate them by parts over the volume S. Through the divergence theorem, one finds for the first equation

$$\begin{aligned}&\int_S u''\left(\frac{\partial}{\partial x}\frac{\partial P'}{\partial a'}+\frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial h'}+\frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial g'}\right)dS\\&=\int_S\frac{\partial}{\partial x}\left(u''\frac{\partial P'}{\partial a'}\right)+\frac{\partial}{\partial y}\left(u''\frac{1}{2}\frac{\partial P'}{\partial h'}\right)+\frac{\partial}{\partial z}\left(u''\frac{1}{2}\frac{\partial P'}{\partial g'}\right)dS\\&\quad-\int_S\frac{\partial u''}{\partial x}\frac{\partial P'}{\partial a'}+\frac{\partial u''}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial h'}+\frac{\partial u''}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial g'}dS\\&=-\int_\sigma u''\left(\frac{\partial P'}{\partial a'}\alpha+\frac{1}{2}\frac{\partial P'}{\partial h'}\beta+\frac{1}{2}\frac{\partial P'}{\partial g'}\gamma\right)d\sigma\\&\quad-\int_S a''\frac{\partial P'}{\partial a'}+h''\frac{1}{2}\frac{\partial P'}{\partial h'}+g''\frac{1}{2}\frac{\partial P'}{\partial g'}dS\end{aligned} \quad (6)$$

It is worth emphasizing that Betti adopts the positive normal vector as directed inward with respect to the body, hence the surface flux resulting from the divergence theorem exhibits the sign minus. Moreover, the strain components a'', g'', and h'' are related to the displacements under the second loading system, namely u'', v'', w'', by the compatibility equation (2). Applying the same procedure for the other two balance equations (along y and z), considering moreover the Cauchy relations over the boundary surface equation (5), one can write

$$\begin{aligned}&-\int_\sigma\left(L'u''+M'v''+N'w''\right)d\sigma\\&-\int_S\frac{\partial P'}{\partial a'}a''+\frac{\partial P'}{\partial b'}b''+\frac{\partial P'}{\partial c'}c''+\frac{\partial P'}{\partial f'}f''+\frac{\partial P'}{\partial g'}g''+\frac{\partial P'}{\partial h'}h''\,dS=0\end{aligned} \quad (7)$$

Analogously, starting from the balance equations relevant to the second loading condition (with apex $''$), one obtains

$$\begin{aligned}&-\int_\sigma\left(L''u'+M''v'+N''w'\right)d\sigma\\&-\int_S\frac{\partial P''}{\partial a''}a'+\frac{\partial P''}{\partial b''}b'+\frac{\partial P''}{\partial c''}c'+\frac{\partial P''}{\partial f''}f'+\frac{\partial P''}{\partial g''}g'+\frac{\partial P''}{\partial h''}h'\,dS=0\end{aligned} \quad (8)$$

Due to the fact that the elastic potential equation (3) is a homogeneous form of the second order, governed by a symmetric

[3] The vector and matrix formulation used nowadays were pioneered by W.R. Hamilton with the theory of quaternions in 1846, further developed by J.J. Sylvester and A. Cayley ten years later, simplified and disseminated by J.W. Gibbs during the 1980s.

[4] Giuseppe Luigi Lagrange (1736–1813), Italian mathematician naturalized French, wrote the two-volume treatise titled "Mécanique Analytique" in 1788–1789, revised in 1811.

[5] Gabriel Lamé (1795–1870), starting from 1820, provided fundamental contributions concerning the curvilinear coordinates, the statistics of structures, and the elasticity theory.

coefficient matrix, the equality holds

$$\frac{\partial P''}{\partial a''}\,a' + \cdots + \frac{\partial P''}{\partial f''}\,f' = \frac{\partial P'}{\partial a'}\,a'' + \cdots + \frac{\partial P'}{\partial f'}\,f'' \tag{9}$$

Hence, one finds

$$\int_\sigma \left(L''u' + M''v' + N''w'\right)d\sigma = \int_\sigma \left(L'u'' + M'v'' + N'w''\right)d\sigma \tag{10}$$

quod erat demonstrandum.

In modern notation (see, e.g., Refs. [13,14]), due to the major symmetry of the stiffness tensor, i.e., $D_{ijkl} = D_{klij}$, one can write $D_{ijkl}\varepsilon'_{ij}\varepsilon''_{kl} = \sigma'_{kl}\varepsilon''_{kl} = D_{ijkl}\varepsilon''_{ij}\varepsilon'_{kl} = \sigma''_{kl}\varepsilon'_{kl}$. The major symmetry of the stiffness tensor is a consequence of the conservativity of the elastic behavior, the work resulting to be path independent: the stress field turns out to be irrotational, due to the existence of a potential and to the Clairaut-Schwarz theorem, since $D_{ijkl} = \partial\sigma_{ij}/\partial\varepsilon_{kl} = \partial^2\omega/\partial\varepsilon_{ij}\partial\varepsilon_{kl}$, being $\omega(\varepsilon_{kl})$ the energy density in classical notation. Since both the loading systems and the corresponding strain and stress states are real, i.e., connected by a causal constraint, they can be regarded, depending on the case, both as statically and kinematically admissible. Applying the principle of virtual work as identity between the set of stresses and forces statically admissible (equilibrated) $\{\sigma'_{ij}, f'_i\}$, and another set of strains and displacements kinematically admissible (or compatible) $\{\varepsilon''_{ij}, s''_i\}$, and then exchanging their role, from the coincidence of the inner virtual works, the equality of the external virtual works follows. Hence, from a mathematical standpoint, the reciprocity theorem is strictly related to the existence of a potential, analogous to Green's identities for harmonic functions largely utilized in those years and mentioned by Betti (see, e.g., Ref. [7], Section IV, page 58). From a mechanical point of view, it stems from the principle of virtual work, widely discussed by Gabrio Piola (1794–1850), see, e.g., Refs. [15–17]. Moreover, it is worth remembering that, in a linear elastic, isotropic, and homogenous elastic body, the volumetric strain turns out to be harmonic, while the displacement components are biharmonic, under the condition that body forces are constant.

In another work included in Ref. [1] (*Sopra l'equazioni di equilibrio dei corpi solidi elastici*, Annali di matematica pura ed applicata, serie II, t. VI, pp. 101–111, Milano, 1874), Betti states the same theorem, with almost identical reasoning, including the volume density of the external forces. The differential linear balance equations at each point of the body, in the presence of forces per unit mass denoted by X', Y', and Z', can be written as

$$\begin{aligned}
\rho X' &= \frac{\partial}{\partial x}\frac{\partial P'}{\partial a'} + \frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial h'} + \frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial g'}\\
\rho Y' &= \frac{\partial}{\partial x}\frac{1}{2}\frac{\partial P'}{\partial h'} + \frac{\partial}{\partial y}\frac{\partial P'}{\partial b'} + \frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial f'}\\
\rho Z' &= \frac{\partial}{\partial x}\frac{1}{2}\frac{\partial P'}{\partial g'} + \frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial f'} + \frac{\partial}{\partial z}\frac{\partial P'}{\partial c'}
\end{aligned} \tag{11}$$

where ρ denotes the material density. In the above equations, the stress components are indicated as the partial derivatives of the potential energy with respect to the strains. Apex $'$ indicates that all the quantities belong to the first loading system. Over the outer boundary σ, positively oriented with the inward normal (having components α, β, γ), the components of the applied surface force are denoted by L', M', and N'. All the quantities relevant to the second system, acting separately on the same body, are denoted by the apex $''$. Let us multiply the lhs of the balance equations (11) ordered by the quantities $u''\,dS$, $v''\,dS$, $w''\,dS$, then integrating over the volume S occupied by the body. Proceeding by parts and applying the divergence theorem, one obtains for the first equation

$$\begin{aligned}
&\int_S u''\left(-\rho X' + \frac{\partial}{\partial x}\frac{\partial P'}{\partial a'} + \frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P'}{\partial h'} + \frac{\partial}{\partial z}\frac{1}{2}\frac{\partial P'}{\partial g'}\right)dS\\
&\quad= -\int_\sigma u''\left(\frac{\partial P'}{\partial a'}\,\alpha + \frac{1}{2}\frac{\partial P'}{\partial h'}\,\beta + \frac{1}{2}\frac{\partial P'}{\partial g'}\,\gamma\right)d\sigma\\
&\qquad - \int_S a''\,\frac{\partial P'}{\partial a'} + h''\,\frac{1}{2}\frac{\partial P'}{\partial h'} + g''\,\frac{1}{2}\frac{\partial P'}{\partial g'}\,dS - \int_S u''\rho X'\,dS
\end{aligned} \tag{12}$$

and analogously for the other equations. Let us repeat the same procedure for the solid subject to the second loading system, i.e., multiplying the balance equations with apices $''$ by the quantities $u'\,dS$, $v'\,dS$, $w'\,dS$ corresponding to the first system, and integrating them over the volume S occupied by the body. Proceeding by parts and applying the divergence theorem, one obtains for the first equation

$$\begin{aligned}
&\int_S u'\left(-\rho X'' + \frac{\partial}{\partial x}\frac{\partial P''}{\partial a''} + \frac{\partial}{\partial y}\frac{1}{2}\frac{\partial P''}{\partial h''} + \frac{\partial}{\partial z}\frac{\partial P''}{2\partial g''}\right)dS\\
&\quad= -\int_\sigma u'\left(\frac{\partial P''}{\partial a''}\,\alpha + \frac{1}{2}\frac{\partial P''}{\partial h''}\,\beta + \frac{1}{2}\frac{\partial P''}{\partial g''}\,\gamma\right)d\sigma\\
&\qquad - \int_S a'\,\frac{\partial P''}{\partial a''} + h'\,\frac{1}{2}\frac{\partial P''}{\partial h''} + g'\,\frac{1}{2}\frac{\partial P''}{\partial g''}\,dS\\
&\qquad - \int_S u'\rho X''\,dS
\end{aligned} \tag{13}$$

and analogously for the other equations. After summing all the contributions, the inner mutual works turn out to be equal, since, as homogeneous quadratic forms, they remain unchanged after a permutation of the strains. Hence, considering σ the equilibrium balance with the surface force densities, we obtain for the external virtual work

$$\begin{aligned}
&\int_\sigma \left(L''u' + M''v' + N''w'\right)d\sigma + \int_S \left(\rho X''u' + \rho Y''v' + \rho Z''w'\right)dS\\
&\quad= \int_\sigma \left(L'u'' + M'v'' + N'w''\right)d\sigma + \int_S \left(\rho X'u'' + \rho Y'v'' + \rho Z'w''\right)dS
\end{aligned} \tag{14}$$

It is worth emphasizing that the same theorem, under more stringent hypotheses, is attributed to James Clerk Maxwell (1831–1879), professor of physics at the University of Cambridge, world-renowned for being the author of the "Treatise on Electricity and Magnetism" published in 1873. In the work [18], dated 1864, Maxwell provided a method to solve hyperstatic frames, i.e., statically indeterminate with redundant constraints, constituted of pinned elastic beams: he recognized that "the extension in BC due to unity of tension along DE is always equal to the extension in DE due to unity of tension in BC." In practice, Maxwell states the symmetry of the influence coefficients relevant to displacements generated by unit forces, utilized in the (equilibrated) force method to specify compatibility equations. By formulae, one can write $\mu_{ij} = \mu_{ji}$, where index i indicates the location of the effect (e.g., a displacement component), while index j denotes the location of the source (i.e., the unit force), μ having dimension $[L/F]$. Moreover, since the two systems of actions dealt with in Maxwell's procedure are both constituted of one unit force, to the equality of the mutual work, one can easily substitute the equality of the displacements of the application points, suitably projected. Maxwell's statement, commonly referred to in the literature as a reduced form of Betti's theorem, is at the basis of the theory of influence lines, as discussed in the last section. In this scenario, the validity of the theorem can be more easily experimentally verified, since the measurement of the displacements is easier than that of work. The experimental verification of Betti's theorem in the formulation of Maxwell was carried out by Lord

John Williams Strutt, 3rd Baron Rayleigh (1842–1919), by means of timber beams subjected to mobile weights.[6]

From the engineering standpoint, the reciprocity theorem answers a question that was not new with reference to elastic beams, see also Ref. [20]. As emphasized by Pugno [12], Leonardo da Vinci (1452–1519) raises a similar question in several pages of the Codex Atlanticus, demonstrating some intuition of the statement. With reference to a simply supported elastic beam, in which a weight positioned in the middle span causes a given deflection, Leonardo asks where a mass, having double or triple weight, should be positioned along the beam to retrieve the same deflection at the midspan. He also asks which weight it should possess, if inversely the location of such a mass is assumed a priori. When recourse is made to the reciprocity theorem, the solution to the above questions is trivial: in any case, the evaluation of the mutual work is necessary, since the weights involved are different.

3 Auxiliary Role of Betti's Principle

In several pages of his works, Betti did interesting applications of the reciprocity principle. For instance, in Ref. [7] (page 323 et seq.), he developed the following didactical exercise for beginners that however also exhibits some practical interest for engineers. If we assume as displacement field an affine function of the coordinates, describing a small rigid body motion governed by six parameters $(\breve{l}, \breve{m}, \breve{n}, \breve{p}, \breve{q}, \breve{r})$ in the three-dimensional space (i.e., three translations and three small rotations), namely,

$$\begin{aligned} u'' &= \breve{l} + \breve{r}y - \breve{q}z \\ v'' &= \breve{m} - \breve{r}x + \breve{p}z \\ w'' &= \breve{n} + \breve{q}x - \breve{p}y \end{aligned} \tag{15}$$

that balance vanishing surface force densities, i.e., $L'' = M'' = N'' = 0$, in the expression of the mutual external virtual work, one has

$$\int_\sigma \left(L'u'' + M'v'' + N'w''\right) d\sigma = 0 \tag{16}$$

and hence

$$\begin{aligned} &\breve{l}\int_\sigma L'\,d\sigma + \breve{m}\int_\sigma M'\,d\sigma + \breve{n}\int_\sigma N'\,d\sigma + \breve{r}\int_\sigma \left(L'y - M'x\right) d\sigma \\ &\quad + \breve{q}\int_\sigma \left(-L'z + M'x\right) d\sigma + \breve{p}\int_\sigma \left(M'z - N'y\right) d\sigma = 0 \quad \forall\, \breve{l}, \breve{m}, \breve{n}, \breve{p}, \breve{q}, \breve{r} \end{aligned} \tag{17}$$

From the above calculations, Betti states that the elastically deformable bodies must also satisfy, as necessary conditions of equilibrium, the cardinal equations of statics (in the number of six in the three-dimensional space): in fact, the integrals above provide, by components, the invariants of the loading system, namely, the resultant force and the resultant moment, which must vanish to ensure the equilibrium regardless of the selected rigid body motion. This exercise represents nothing other than the principle of virtual work, written for a compatible set of displacements and a self-equilibrated system of loading. It is worth noting that $\breve{p}$, $\breve{q}$, and $\breve{r}$ represent the components of the axial vector of a skew-symmetric matrix, describing small rotations: the corresponding displacements can be obtained by the matrix product (rows by columns) of that matrix with the position vector, or equivalently by the cross product of such a rotation vector with the position vector. In this respect, it is worth emphasizing that the principle of virtual work was utilized since Greek antiquity to assess the equilibrium conditions of rigid bodies entering mechanisms of technological interest, such as scales or pulleys, see, e.g., Refs. [21,22].

Another important section of Betti's treatise is titled "Dilatation of any element belonging to an isotropically elastic body, subjected to forces acting exclusively on the surface"[7] : the question raised here concerns the evaluation of the volumetric strain at any point of a homogeneous and isotropic linearly elastic body, as a function of the loading density applied on its boundary. To this purpose, Betti conjectures as displacement field at the equilibrium the superposition of two contributions

$$u'' = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \xi, \quad v'' = \frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \eta, \quad w'' = \frac{\partial}{\partial z}\left(\frac{1}{r}\right) + \zeta \tag{18}$$

where the role of the second field, denoted by Greek letters and acting as a "corrector," will be discussed later. In Betti's notation, symbol $r = \|\mathbf{x} - \mathbf{x}'\| = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}$ denotes the relative Euclidean distance between the generic point with coordinates $\mathbf{x}$ and that with coordinates $\mathbf{x}'$, being $1/r$ singular when $\mathbf{x} \to \mathbf{x}'$. Assuming that the point with coordinates $\mathbf{x}'$ belongs to the interior of the domain occupied by the body, denoted by symbol S, whilst $\mathbf{x}$ can also vary over its boundary (belonging to the closure $\bar{S}$), the displacement components corresponding to the first contribution, i.e., $\partial r^{-1}/\partial x$, etc., are bounded and continuous with their derivatives in the whole region deprived of a sphere centered at $\mathbf{x}'$, say s, with an infinitesimal radius. We can denote this region by $S' = S - s \subset S$, enclosed by the disjoint surfaces $\sigma = \partial S$ and σ', the union of which constitutes the frontier of S'. The first contribution in Eq. (18) is the gradient of a potential: function r^{-1} constitutes the fundamental solution of the Laplace equation for the three-dimensional case and coincides with the well-known Coulomb (central, radial) scalar potential, generated by a point electric charge and vanishing at infinity, its gradient being the electric vector field. The displacement expressed by the gradient of the potential r^{-1} generates in the body a null volumetric strain, corresponding to a vanishing divergence that coincides indeed with the Laplacian of r^{-1}. Also, the Laplacian of each one of the components of the potential gradient vanishes, as it can easily be recognized by the Clairaut-Schwarz theorem when commuting the differential operators: hence, the components of the first contribution in Eq. (18) result in harmonic functions. By formulae, one has

$$\begin{aligned} \Theta(\mathrm{grad}\, r^{-1}) &= \mathrm{div}\left(\mathrm{grad}\, r^{-1}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) \\ &= \nabla^2\left(\frac{1}{r}\right) = 0 \\ \nabla^2\left(\frac{\partial}{\partial x}\frac{1}{r}\right) &= \frac{\partial}{\partial x}\nabla^2\left(\frac{1}{r}\right) = 0, \quad \nabla^2\left(\frac{\partial}{\partial y}\frac{1}{r}\right) = 0, \quad \nabla^2\left(\frac{\partial}{\partial z}\frac{1}{r}\right) = 0 \end{aligned} \tag{19}$$

In Betti's Ansatz, the first contribution constitutes a field, at the same time, irrotational (i.e., rotgrad $\cdot$ =0) and soleinodal. Exclusively, the second contribution to the solution field, expressed by the components ξ, η, and ζ, contributes to the Laplacian and to the divergence of the displacement field, i.e.,

[6]In the two-volume work "The Theory of sound," dated 1877–1878, Lord Rayleigh extended the formulation of Maxwell to vibrations and acoustics, proving that in stationary conditions, the source and receiver location can be interchanged (see Ref. [19], Chapter IV, Vibrating systems in general, page 68 et seq.). Furthermore, in a late edition of the same treatise dated 1894–1896, establishing an analogy between pairs of mechanical and physical quantities, namely, between the sound pressure and the voltage, the fluid velocity and the electric current, Rayleigh also provided an extension to electric currents in passive networks: this last principle is referred to in the literature as Rayleigh-Carson, as a special case of Lorentz reciprocity involving electromagnetic and electric fields.

[7]"Dilatazione di un elemento qualunque di un corpo elastico isotropo, sotto l'azione di forze che agiscono soltanto alla superficie" ([7], Section VIII, pages 335 et seq.).

$\nabla^2 u'' = \nabla^2 \xi$, $\nabla^2 v'' = \nabla^2 \eta$, $\nabla^2 w'' = \nabla^2 \zeta$, and $\Theta'' = \partial\xi/\partial x + \partial\eta/\partial y + \partial\zeta/\partial z$, and therefore can satisfy the Navier-Cauchy equations, i.e.,

$$\begin{aligned}(2\lambda+\mu)\frac{\partial}{\partial x}\Theta'' + \mu\nabla^2\xi = 0\\ (2\lambda+\mu)\frac{\partial}{\partial y}\Theta'' + \mu\nabla^2\eta = 0\\ (2\lambda+\mu)\frac{\partial}{\partial z}\Theta'' + \mu\nabla^2\zeta = 0\end{aligned} \tag{20}$$

where body forces have been neglected. In other words, in the partition equation (18), the displacement field with components ξ, η, ζ permits to satisfy Eq. (20) within the bulk (also in the presence of volume force densities), while the addends relevant to the gradient of the potential belong to the kernel of the Navier operator: on the contrary, both the displacement fields contribute to the equilibrium conditions on the boundary (Eq. (5)), although the potential gradient generates purely deviatoric strain and stress fields. It should not surprise the reader that, in Betti's work, the second Lamé modulus λ appears multiplied by two (in the Navier equations as in the constitutive relations), due to the definition that Betti adopts for the potential energy density (Eq. (3)), without the coefficient $1/2$.[8]

In the volume S', deprived of an infinitesimal sphere centered at the singular point $\mathbf{x}'$, let us define the displacement field with components u', v', w', specifying the equilibrium configuration under force densities acting on the outer surfaces σ and σ', denoted by the symbols L', M', N', and L_1', M_1', N_1', respectively. In these passages, we have slightly modified the notation used by Betti, avoiding using symbols X, Y, and Z that already indicate the forces per unit mass: moreover, in what follows, pedex 1 will denote the force density components acting on the infinitesimal spherical surface σ' centered at $\mathbf{x}'$. Let us notice that, on the boundary surface σ', the normal vector pointing inward the solid body results outward with respect to the sphere: the force density prescribed on σ' is equal and opposite to the components of the traction vector acting on it from inside the body, and hence indicated with the sign $-$. One can express the boundary equilibrium equation on σ' in terms of the displacement components u', v', w':

$$\begin{aligned}-L_1' &= 2\left(\lambda\Theta' + \mu\frac{\partial u'}{\partial x}\right)\alpha + \mu\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\beta + \mu\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)\gamma\\ -M_1' &= \mu\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right)\alpha + 2\left(\lambda\Theta' + \mu\frac{\partial v'}{\partial y}\right)\beta + \mu\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)\gamma\\ -N_1' &= \mu\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)\alpha + \mu\left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right)\beta + 2\left(\lambda\Theta' + \mu\frac{\partial w'}{\partial z}\right)\gamma\end{aligned} \tag{21}$$

and on σ

$$\begin{aligned}L' &= 2\left(\lambda\Theta' + \mu\frac{\partial u'}{\partial x}\right)\alpha + \mu\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)\beta + \mu\left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x}\right)\gamma\\ M' &= \mu\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right)\alpha + 2\left(\lambda\Theta' + \mu\frac{\partial v'}{\partial y}\right)\beta + \mu\left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)\gamma\\ N' &= \mu\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)\alpha + \mu\left(\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial z}\right)\beta + 2\left(\lambda\Theta' + \mu\frac{\partial w'}{\partial z}\right)\gamma\end{aligned} \tag{22}$$

being λ and μ the Lamé moduli, and Θ the volumetric strain. The displacement field in Eq. (18) with components u'', v'', w'', resulting from the superposition of the gradient of the potential r^{-1} and of the field ξ, η,ζ, specifies the equilibrium configuration under surface force densities applied on σ and σ', that due to the linearity of the boundary equilibrium equation can be expressed in components as $(L''+L^0)$, $(M''+M^0)$, $(N''+N^0)$, and $(L_1''+L_1^0)$, $(M_1''+M_1^0)$, $(N_1''+N_1^0)$, respectively. Betti's theorem implies the equality of the mutual (external) virtual works between the displacement field specified above and their surface loading, i.e.,

$$\begin{aligned}&\int_\sigma\left\{L'\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\xi\right]+M'\left[\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\eta\right]+N'\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)+\zeta\right]\right\}d\sigma\\ &\quad+\int_{\sigma'}\left\{L_1'\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\xi\right]+M_1'\left[\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\eta\right]+N_1'\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)+\zeta\right]\right\}d\sigma'\\ &=\int_\sigma\left\{\left[L''+L^0\right]u'+\left[M''+M^0\right]v'+\left[Z''+Z^0\right]w'\right\}d\sigma\\ &\quad+\int_{\sigma'}\left\{\left[L_1''+L_1^0\right]u'+\left[M_1''+M_1^0\right]v'+\left[N_1''+N_1^0\right]w'\right\}d\sigma'\end{aligned} \tag{23}$$

Among the above contributions, let us notice that the surface integrals of bounded and continuous quantities over the sphere σ' with an infinitesimal radius vanish, namely, for $r \to 0$, one has

$$\int_{\sigma'}\left\{L_1'\xi + M_1'\eta + N_1'\zeta\right\}d\sigma' = \int_{\sigma'}\left\{L_1^0 u' + M_1^0 v' + N_1^0 w'\right\}d\sigma' \to 0 \tag{24}$$

Moreover, let us consider in Eq. (23) the last integral on the spherical surface σ' (with infinitesimal radius): the surface force density, with components L_1'', M_1'', N_1'', due to the reciprocity formulation, it is balanced by the displacement field that is the gradient of r^{-1} and induces a null volumetric strain. According to the boundary equilibrium equations (21), the first component of this distribution can be expressed as

$$\begin{aligned}-L_1'' &= 2\mu\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{1}{r}\right)\alpha + \mu\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{1}{r} + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{1}{r}\right)\beta\\ &\quad+\mu\left(\frac{\partial}{\partial z}\frac{\partial}{\partial x}\frac{1}{r} + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\frac{1}{r}\right)\gamma = 2\mu\frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\alpha + 2\mu\frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\beta\\ &\quad+2\mu\frac{\partial}{\partial z}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\gamma\end{aligned} \tag{25}$$

on the basis of Clairaut-Schwarz theorem. Following the same path, the remaining components can be expressed in turn as

$$\begin{aligned}-M_1'' &= 2\mu\frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\alpha + 2\mu\frac{\partial}{\partial y}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\beta + 2\mu\frac{\partial}{\partial y}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\gamma\\ -N_1'' &= 2\mu\frac{\partial}{\partial z}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\alpha + 2\mu\frac{\partial}{\partial z}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\beta + 2\mu\frac{\partial}{\partial z}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\gamma\end{aligned} \tag{26}$$

Denoting by p the normal to the boundary surface, that over the infinitesimal inner boundary σ' coincides with the direction of the position vector from $\mathbf{x}'$, and the normal director cosines can be expressed as $\alpha = dx/dp$, $\beta = dy/dp$, and $\gamma = dz/dp$. Equations (25) and (26) can be written in a compact form as

$$-L_1'' = 2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial x}\left(\frac{1}{r}\right),\quad -M_1'' = 2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial y}\left(\frac{1}{r}\right),\quad -N_1'' = 2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) \tag{27}$$

It is worth highlighting the meaning of the above expression. In fact, as for the displacements generated by the potential gradient, the equilibrium boundary condition (Eq. (21)) on σ' becomes a Neumann condition involving the normal derivative of the unknown field: in general, this condition does not hold, since the presence of the volumetric strain makes the expression more convoluted.

Taking into account the vanishing addends of Eq. (24) and the relations just deduced for the surface force densities (Eq. (27)), Equation (23) can be arranged grouping at the l.h.s. the integrals

[8] Actually, the first to introduce the coefficient $\frac{1}{2}$ multiplying the kinetic energy was Gaspard-Gustave de Coriolis (1792–1843) in his work dated 1829 and titled "Du calcul de l'effet des machines," and for the elastic energy, George Green (1793–1841), probably around 1837. In his fundamental treatise titled "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," dated 1828, Green coined the term potential.

over σ' that include functions with a singularity in $\mathbf{x}'$ as follows:

$$\int_{\sigma'}\left\{2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)u'+2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)v'+2\mu\frac{\partial}{\partial p}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)w'\right\}d\sigma'$$
$$+\int_{\sigma'}\left\{L_1'\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+M_1'\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+N_1'\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\}d\sigma'$$
$$=\int_{\sigma}\left\{\left(L''+L^0\right)u'+\left(M''+M^0\right)v'+\left(N''+N^0\right)w'\right\}d\sigma$$
$$-\int_{\sigma}\left\{L'\left[\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\xi\right]+M'\left[\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\eta\right]+N'\left[\frac{\partial}{\partial z}\left(\frac{1}{r}\right)+\zeta\right]\right\}d\sigma \tag{28}$$

At the r.h.s., we recognize all the surface integrals evaluated on σ, which we can try to simplify. With reference to the last integral, let us notice that, if we multiply L' by $(\partial/\partial x)(r^{-1})$, M' by $(\partial/\partial y)(r^{-1})$, and N' by $(\partial/\partial z)(r^{-1})$, on the basis of Eq. (22), one can write on σ

$$\begin{aligned}
L'\frac{\partial}{\partial x}\left(r^{-1}\right)&=-2\lambda\Theta'\frac{\partial r^{-1}}{\partial x}\frac{dx}{dr}-\mu\frac{du'}{dr}\frac{\partial r^{-1}}{\partial x}-\mu\left(\frac{\partial u'}{\partial x}\frac{\partial r^{-1}}{\partial x}+\frac{\partial v'}{\partial y}\frac{\partial r^{-1}}{\partial y}+\frac{\partial w'}{\partial x}\frac{\partial r^{-1}}{\partial z}\right)\frac{dx}{dr}\\
M'\frac{\partial}{\partial y}\left(r^{-1}\right)&=-2\lambda\Theta'\frac{\partial r^{-1}}{\partial y}\frac{dy}{dr}-\mu\frac{dv'}{dr}\frac{\partial r^{-1}}{\partial y}-\mu\left(\frac{\partial u'}{\partial y}\frac{\partial r^{-1}}{\partial x}+\frac{\partial v'}{\partial y}\frac{\partial r^{-1}}{\partial y}+\frac{\partial w'}{\partial y}\frac{\partial r^{-1}}{\partial z}\right)\frac{dy}{dr}\\
N'\frac{\partial}{\partial z}\left(r^{-1}\right)&=-2\lambda\Theta'\frac{\partial r^{-1}}{\partial z}\frac{dz}{dr}-\mu\frac{dw'}{dr}\frac{\partial r^{-1}}{\partial z}-\mu\left(\frac{\partial u'}{\partial z}\frac{\partial r^{-1}}{\partial x}+\frac{\partial v'}{\partial z}\frac{\partial r^{-1}}{\partial y}+\frac{\partial w'}{\partial z}\frac{\partial r^{-1}}{\partial z}\right)\frac{dz}{dr}
\end{aligned} \tag{29}$$

For the above equations, we have utilized the chain of equalities:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\frac{dy}{dr}=\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\frac{dx}{dr},\quad \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\frac{dz}{dr}=\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\frac{dy}{dr},\quad \text{etc.}\\
&\frac{d\cdot}{dr}=\frac{\partial\cdot}{\partial x}\frac{dx}{dr}+\frac{\partial\cdot}{\partial y}\frac{dy}{dr}+\frac{\partial\cdot}{\partial z}\frac{dz}{dr}
\end{aligned} \tag{30}$$

By adding side by side Eq. (29), one obtains

$$L'\frac{\partial r^{-1}}{\partial x}+M'\frac{\partial r^{-1}}{\partial y}+N'\frac{\partial r^{-1}}{\partial z}=-2\lambda\Theta'\frac{dr^{-1}}{dr}-2\mu\left(\frac{du'}{dr}\frac{\partial r^{-1}}{\partial x}+\frac{dv'}{dr}\frac{\partial r^{-1}}{\partial y}+\frac{dw'}{dr}\frac{\partial r^{-1}}{\partial z}\right) \tag{31}$$

The reader might wonder whether the integral on σ' of the expression at the r.h.s. is convergent, including powers of r at the denominator. Now, one of the most famous results of potential theory due to Gauss is that the following surface integral is convergent, and when $r\to 0$, one has

$$\int_{\sigma'}d\sigma'\Theta'\frac{d\,r^{-1}}{dr}=-4\pi\Theta \tag{32}$$

Symbol Θ denotes herein the volumetric strain Θ' (relevant to the field u', v', w') evaluated at the point $\mathbf{x}'$, at the center of the sphere σ'. The above result can be easily proven when considering that the surface element of a sphere with radius r equals $r^2\sin\vartheta d\vartheta d\varphi=r^2\,d\varpi$, being ϑ and φ the polar and the azimuthal angle ($0\le\vartheta\le\pi, 0\le\varphi\le 2\pi$), while $d\varpi$ indicates the corresponding area element for the unit sphere ($r=1$). The above equality can also be deduced as a special case of Green's identities discussed later, letting $r\to 0$. At this point, Betti utilizes the following equalities on the infinitesimal spherical surface σ' (i.e., with $r\to 0$):

$$\begin{aligned}
\int_{\sigma'}d\sigma'\left\{\frac{du'}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)-u'\frac{d}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right\}&=-4\pi\left.\frac{\partial u'}{\partial x}\right|_{\mathbf{x}'}\\
\int_{\sigma'}d\sigma'\left\{\frac{dv'}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)-u'\frac{d}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right\}&=-4\pi\left.\frac{\partial v'}{\partial y}\right|_{\mathbf{x}'}\\
\int_{\sigma'}d\sigma'\left\{\frac{dw'}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)-u'\frac{d}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\}&=-4\pi\left.\frac{\partial w'}{\partial z}\right|_{\mathbf{x}'}
\end{aligned} \tag{33}$$

It is worth noting that the normal to the spherical surface σ' is parallel to the position vector starting from the center, hence $d/dr=d/dn$: the above partial derivatives represent normal derivatives. In addition, within the above integrals, we are convolving a single displacement component and its normal derivative not just with the potential function, but with its partial derivative in the direction of the displacement, i.e., u' with $\partial/\partial x(r^{-1})$, v' with $\partial/\partial y(r^{-1})$, and w' with $\partial/\partial z(r^{-1})$. Consistently, at the r.h.s., we recognize, multiplied by -4π, not the displacement component evaluated at the center $\mathbf{x}'$, but its partial derivative in the same direction. This circumstance will be clarified at the end of this section, when the properties of the Green's function will be addressed. The proof of Eq. (33) adduced by Betti may appear cumbersome and somewhat obscure to the modern reader. In fact, Betti observes that, on σ', one can write

$$\int_{\sigma'}d\sigma'\left\{\frac{du'}{dr}\frac{\partial r^{-1}}{\partial x}-u'\frac{d}{dr}\frac{\partial r^{-1}}{\partial x}\right\}=\int_{\sigma'}d\sigma'\left\{\frac{du'}{dr}\frac{\alpha}{r^2}-u'\frac{d}{dr}\left(\frac{\alpha}{r^2}\right)\right\}=\int_{\sigma'}d\sigma'\frac{1}{r^4}\frac{d}{dr}\left(u'\alpha r^2\right) \tag{34}$$

where $\alpha=-x/r$, $\beta=-y/r$, and $\gamma=-z/r$ are the director cosines of the normal to the outer surface, oriented inward. The above relation can be easily proved by passing to spherical coordinates, being $x=r\cos\vartheta\cos\varphi$, $x=r\cos\vartheta\sin\varphi$, and $z=r\sin\vartheta$.

Applying the divergence theorem to the horizontal strain over the three-dimensional region s, one obtains

$$\int_s ds\frac{\partial u'}{\partial x}=-\int_{\sigma'}d\sigma' u'\alpha=-r^2\int_{\varpi}d\varpi u'\alpha \tag{35}$$

being $d\varpi$ the spherical surface element with unit radius, and the sign $-$ is due to the orientation of the normal preferred by Betti. At this point, the reasoning of Betti could result misleading for the reader, since the integrand function (possibly nonlinear) must always be evaluated over the sphere with radius r, although the surface element could be scaled proportionally to r^2 (being $r^2\sin\vartheta$ the Jacobian determinant) and the radius is not included among the integration variables of the surface integral. Clearly, to compute the volume (triple) integral, one also has to integrate with respect to variable r, being the volume element expressed as $ds=r^2\sin\vartheta\, dr d\vartheta d\varphi$, obtaining

$$\int_0^r\tilde{r}^2d\tilde{r}\int_{\varpi}d\varpi\frac{\partial u'}{\partial x}=-\int_{\sigma'}d\sigma' u'\alpha=-\int_{\varpi}d\varpi u'\alpha r^2 \tag{36}$$

Differentiating both sides with respect to r and dividing by r^2, one can write

$$\int_{\varpi} d\varpi \frac{\partial u'}{\partial x} = 4\pi \frac{\partial u'}{\partial x}\bigg|_{\mathbf{x}'} = -\int_{\varpi} d\varpi \frac{1}{r^2}\frac{d}{dr}(u'\alpha r^2) = -\int_{\sigma'} d\sigma' \frac{1}{r^4}\frac{d}{dr}(u'\alpha r^2) \tag{37}$$

The first equality strictly holds if the function in point is harmonic, hence coincident with its surface or volume average: this hypothesis was not assumed explicitly by Betti, and, however, it is the volumetric strain to be harmonic, not the single displacement derivative entering it. Without this assumption, the function at the right-hand side, i.e., $4\pi(\partial u'/\partial x)$, has to be evaluated at a point over the surface, and not at the center of the sphere. In any case, this could be a light fault since it is worth remembering that we are dealing with the case $r \to 0$. Finally, one obtains

$$\int_{\sigma'} d\sigma' \left\{ \frac{du'}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - u'\frac{d}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) \right\} = -4\pi \frac{\partial u'}{\partial x}\bigg|_{\mathbf{x}'} \tag{38}$$

Let us observe that with superimposing expressions already analyzed, Eqs. (32) and (33), one can write

$$\begin{aligned}
&L'\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + M'\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + N'\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + 2\mu\left\{\frac{du'}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{dv'}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \frac{dw'}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\} \\
&\quad - 2\mu\left\{\frac{du'}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) - u'\frac{d}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right\} - 2\mu\left\{\frac{dv'}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) - v'\frac{d}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)\right\} \\
&\quad - 2\mu\left\{\frac{dw'}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right) - w'\frac{d}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\} \\
&\ = L'\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + M'\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + N'\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + 2\mu\left\{u'\frac{d}{dr}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + v'\frac{d}{dr}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + w'\frac{d}{dr}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\}
\end{aligned} \tag{39}$$

It is worth noting that the sum of the r.h.s of Eq. (33) provides $-4\pi\Theta$, where the volumetric strain evaluated at $\mathbf{x}'$ equals $\Theta = \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z}\right)|_{\mathbf{x}'}$: hence, the sum of the three curly brackets just before the first equality in Eq. (39), multiplied by -2μ, provides $8\pi\mu\Theta$. Integrating Eq. (31) over the surface σ' and considering previous results, one obtains

$$\begin{aligned}
&\int_{\sigma'} d\sigma'\left[L'\frac{\partial r^{-1}}{\partial x} + M'\frac{\partial r^{-1}}{\partial y} + N'\frac{\partial r^{-1}}{\partial z} + 2\mu\left(u'\frac{d}{dr}\frac{\partial r^{-1}}{\partial x} + v'\frac{d}{dr}\frac{\partial r^{-1}}{\partial y} + w'\frac{d}{dr}\frac{\partial r^{-1}}{\partial z}\right)\right] \\
&\quad = 8\pi(\lambda+\mu)\Theta
\end{aligned} \tag{40}$$

From Eq. (23), one can finally compute the volumetric strain at point $\mathbf{x}'$

$$\begin{aligned}
\Theta = &-\frac{1}{8\pi(\lambda+\mu)}\int_{\sigma}\left\{\left[L'\left(\frac{\partial r^{-1}}{\partial x} + \xi\right) + M'\left(\frac{\partial r^{-1}}{\partial y} + \eta\right) + N'\left(\frac{\partial r^{-1}}{\partial z} + \zeta\right)\right]\right. \\
&\left. - \left(L'' + L^0\right)u' - \left(M'' + M^0\right)v' - \left(N'' + N^0\right)w'\right\}d\sigma
\end{aligned} \tag{41}$$

The displacement field with components u', v', and w', appearing in the above integral and generated by the contact pressures applied on σ, with components L', M', and N', is unknown. The same occurs for the fields ξ, η, and ζ, satisfying the volume of the Navier equations (20) and balancing the external force density L^0, M^0, and N^0 on σ. On the basis of Eq. (27), the surface force density over the boundary, which corresponds to the gradient of the potential r^{-1}, can be expressed in the compact form $L'' = -2\mu(d/dp)(\partial r^{-1}/\partial x)$, etc., being p the surface normal. Hence, if we are able to select the "correction" displacements ξ, η, and ζ in such a way that they satisfy the Navier equations within the bulk and balance the outer force distributions due to the gradient of potential, obtaining $L'' + L^0 = 0$, $M'' + M^0 = 0$, and $N'' + N^0 = 0$, the three addends in Eq. (41) multiplying the unknown field u', v', and w' disappear. Under these conditions (indicated by a tilde the field fulfilling the above conditions), the final formula becomes

$$\Theta(\mathrm{R}) = -\frac{1}{8\pi(\lambda+\mu)}\int_{\sigma}\left\{\left[L'\left(\frac{\partial r_{\mathrm{RQ}}^{-1}}{\partial x} + \tilde{\xi}(\mathrm{Q})\right) + M'\left(\frac{\partial r_{\mathrm{RQ}}^{-1}}{\partial y} + \tilde{\eta}(\mathrm{Q})\right) + N'\left(\frac{\partial r_{\mathrm{RQ}}^{-1}}{\partial z} + \tilde{\zeta}(\mathrm{Q})\right)\right]\right\}d\sigma_{\mathrm{Q}} \tag{42}$$

For the sake of clarity, with respect to the original Betti's presentation, we have entered symbols $\mathrm{R} = \{\mathbf{x}'\}$ and $\mathrm{Q} = \{\mathbf{x}\}$, underlying the fact that the potential plays the role of Green's symmetric kernel $G(\mathrm{R},\mathrm{Q}) = G(\mathrm{R}-\mathrm{Q})$ and that the integration is carried out with respect to the argument Q alone over the boundary σ. To compute the displacement field with components $\tilde{\xi}$, $\tilde{\eta}$, and $\tilde{\zeta}$, Betti suggests as a general method to expand in spherical harmonics the prescribed loading terms and the unknowns, specifying the constraints they have to satisfy, term by term, to fulfill the equilibrium equations.

The conceptual path developed by Betti may appear to the reader at times obscure, also because it was carried out by components. As an alternative, one can deduce directly the volumetric strain Θ, which is a harmonic function for a linearly elastic isotropic body, using Green's identities. In potential theory, see, e.g., Refs. [23,24], it is well known that if U and V are functions and sufficiently regular and defined within a domain S with boundary σ, as a consequence of the divergence theorem, the following equality holds:

$$\int_{\sigma}\left\{U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right\}d\sigma = \int_{S}\left\{U\nabla^2 V - V\nabla^2 U\right\}dS = 0 \tag{43}$$

where $\mathbf{n}$ is the outward normal to the boundary, and for the directional derivative, one can write $\partial V/\partial n = \nabla V \cdot \mathbf{n}$. If $G(\mathrm{R},\mathrm{Q})$ is the Green's function for the three-dimensional Laplace operator, i.e., the response of the system to an impulse $\nabla^2_{\mathrm{R}} G(\mathrm{R},\mathrm{Q}) = \delta^3(\mathrm{R}-\mathrm{Q})$, where δ^3 is the

Dirac distribution in the three-dimensional space, one can write

$$\int_\sigma \left\{ U(\mathrm{Q}) \frac{\partial}{\partial_{\mathrm{Q}} n} G(\mathrm{R}, \mathrm{Q}) - G(\mathrm{R}, \mathrm{Q}) \frac{\partial U}{\partial_{\mathrm{Q}} n} \right\} d\sigma_{\mathrm{Q}} \\ = \int_S \left\{ U(\mathrm{Q}) \nabla^2_{\mathrm{Q}} G(\mathrm{R}, \mathrm{Q}) - G(\mathrm{R}, \mathrm{Q}) \nabla^2_{\mathrm{Q}} U \right\} dS_{\mathrm{Q}} = U(\mathrm{R}) \\ - \int_S G(\mathrm{R}, \mathrm{Q}) \nabla^2_{\mathrm{Q}} U \, dS_{\mathrm{Q}} \tag{44}$$

resulting, after convolution with the Dirac delta in the spirit of distributions,

$$\int_S U(\mathrm{Q}) \nabla^2_{\mathrm{Q}} G(\mathrm{R}, \mathrm{Q}) \, dS_{\mathrm{Q}} = \int_S U(\mathrm{Q}) \delta^3(\mathrm{R} - \mathrm{Q}) \, dS_{\mathrm{Q}} = U(\mathrm{R}) \tag{45}$$

Hence, the value of U at the point R can be computed as follows:

$$U(\mathrm{R}) = \int_S G(\mathrm{R}, \mathrm{Q}) \nabla^2_{\mathrm{Q}} U \, dS_{\mathrm{Q}} \\ + \int_\sigma \left\{ U(\mathrm{Q}) \frac{\partial}{\partial_{\mathrm{Q}} n} G(\mathrm{R}, \mathrm{Q}) - G(\mathrm{R}, \mathrm{Q}) \frac{\partial U}{\partial_{\mathrm{Q}} n} \right\} d\sigma_{\mathrm{Q}} \tag{46}$$

The Green's function for the three-dimensional Laplace operator is

$$G(\mathrm{R}, \mathrm{Q}) = G(\mathrm{R} - \mathrm{Q}) = \frac{-1}{4\pi \|\mathbf{x}_{\mathrm{R}} - \mathbf{x}_{\mathrm{Q}}\|} = -\frac{1}{4\pi r_{\mathrm{RQ}}} \tag{47}$$

and is symmetric with respect to its arguments, i.e., $G(\mathrm{R}, \mathrm{Q}) = G(\mathrm{Q}, \mathrm{R}) \, \forall \, \mathrm{R}, \mathrm{Q}$. This property is often referred to as the reciprocity of the Green's function. The sign $-$ is required to ensure consistency of the definition of G provided above with the divergence theorem and the flux sign, being $\nabla^2 G = \mathrm{div}(\nabla G)$. It is worth noting that, in Eq. (43), a change of sign in the normal vector implies a change of sign of the two surface contributions at the l.h.s. (including normal derivatives), while a change of sign in the Green's function implies a change of sign of both the volume contributions at the r.h.s., also due to the linearity of the Laplace operator.

An analogous representation formula can be provided for the first-order partial derivatives of function U at point R, differentiating the kernel under the integral with respect to the free variable R not affected by the integration. One obtains

$$\nabla_{\mathrm{R}} U(\mathrm{R}) = \int_S \nabla_{\mathrm{R}} G(\mathrm{R}, \mathrm{Q}) \nabla^2_{\mathrm{Q}} U \, dS_{\mathrm{Q}} \\ + \int_\sigma \left\{ U(\mathrm{Q}) \frac{\partial}{\partial_{\mathrm{Q}} n} \nabla_{\mathrm{R}} G(\mathrm{R}, \mathrm{Q}) - \nabla_{\mathrm{R}} G(\mathrm{R}, \mathrm{Q}) \frac{\partial U}{\partial_{\mathrm{Q}} n} \right\} d\sigma_{\mathrm{Q}} \tag{48}$$

where we have commuted the nabla operator with the directional derivative.

If a Green's function can be provided, vanishing on the boundary σ, itself or its normal derivative, the surface integral can be further simplified. For instance, one can assume

$$\widetilde{G}(\mathrm{R}, \mathrm{Q}) = \frac{-1}{4\pi \|\mathbf{x}_{\mathrm{R}} - \mathbf{x}_{\mathrm{Q}}\|} + \phi(\mathrm{R}, \mathrm{Q}) \tag{49}$$

where function ϕ is selected in such a way to provide the same value of G (or of its normal derivative) over the boundary, with opposite sign. Moreover, when U is harmonic within the domain, i.e., $\nabla^2 U = 0$, the volume integral disappears: hence the values of U and of its gradient $\nabla_{\mathrm{R}} U$ in the interior domain are completely known once their values on the outer boundary have been prescribed. Assuming Green's function as in Eq. (47), one obtains for the function U and its gradient

$$4\pi U(\mathrm{R}) = \int_S \left(\frac{-1}{r_{\mathrm{RQ}}} \right) \nabla^2_{\mathrm{Q}} U \, dS_{\mathrm{Q}} \\ + \int_\sigma \left\{ U(\mathrm{Q}) \frac{\partial}{\partial_{\mathrm{Q}} n} \left(\frac{-1}{r_{\mathrm{RQ}}} \right) - \left(\frac{-1}{r_{\mathrm{RQ}}} \right) \frac{\partial U}{\partial_{\mathrm{Q}} n} \right\} d\sigma_{\mathrm{Q}} \\ 4\pi \nabla U(\mathrm{R}) = \int_S \nabla_{\mathrm{R}} \left(\frac{-1}{r_{\mathrm{RQ}}} \right) \nabla^2_{\mathrm{Q}} U \, dS_{\mathrm{Q}} \\ + \int_\sigma \left\{ U(\mathrm{Q}) \frac{\partial}{\partial_{\mathrm{Q}} n} \nabla_{\mathrm{R}} \left(\frac{-1}{r_{\mathrm{RQ}}} \right) - \nabla_{\mathrm{R}} \left(\frac{-1}{r_{\mathrm{RQ}}} \right) \frac{\partial U}{\partial_{\mathrm{Q}} n} \right\} d\sigma_{\mathrm{Q}} \tag{50}$$

It is worth noting that several expressions deduced by Betti and commented above were referred to an infinitesimal sphere σ' inner to the integration domain and were intended to prove the convergence of the singular integrals letting the radius r tend to zero. Once this property is proved, the Greens' identities can be applied to any couple of fields sufficiently regular, over any closed domain with a smooth boundary. A special scenario occurs when the singular point (or source point) lies on the frontier: in this case, the integral is improper and the value of the function at those points has to be multiplied by a coefficient of $1/2$ if the surface is regular at that point.

On the basis of the Green's identities outlined above in a modern formulation, let us try to retrieve Eq. (33) utilized by Betti over the infinitesimal sphere s centered in $\mathbf{x}'$, with boundary σ' . In particular, let us particularize Eq. (50) to the partial derivative with respect to x (i.e., the horizontal strain ε_{xx}), namely,

$$4\pi \left. \frac{\partial u}{\partial x} \right|_{\mathbf{x}'} = \int_s \frac{\partial}{\partial x} \left(\frac{-1}{r} \right) \nabla^2 u \, ds + \\ \int_{\sigma'} \left\{ u \frac{\partial}{\partial n} \frac{\partial}{\partial x} \left(\frac{-1}{r} \right) - \frac{\partial}{\partial x} \left(\frac{-1}{r} \right) \frac{\partial u}{\partial n} \right\} d\sigma' \\ = \underbrace{- \int_s \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \nabla^2 u \, ds}_{=(\star)} + \int_{\sigma'} \left\{ -u \frac{\partial}{\partial r} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \frac{\partial u}{\partial r} \right\} d\sigma' \tag{51}$$

where we have set $d/dr = d/dn$, since the outward normal vector $\mathbf{n}$ was used herein. Integrating by parts the first contribution, and passing to the boundary, we obtain

$$(\star) = - \int_s \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \nabla^2 u \, ds = - \int_{\sigma'} \left(\frac{1}{r} \nabla^2 u \right) n_x \, d\sigma' + \int_s \left(\frac{1}{r} \right) \nabla^2 \frac{\partial u}{\partial x} \, ds \tag{52}$$

Both these terms vanish when $r \to 0$, being the surface element and the volume element higher order infinitesimals with respect to the integrands. Due to the choice of orienting the normal inward, the directional derivative must change sign, finally obtaining

$$-4\pi \left. \frac{\partial u}{\partial x} \right|_{\mathbf{x}'} = \int_{\sigma'} \left\{ -u \frac{\partial}{\partial r} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \frac{\partial}{\partial x} \left(\frac{1}{r} \right) \frac{\partial u}{\partial r} \right\} d\sigma' \tag{53}$$

as in Eq. (33).

4 Remarks Between Past and Future

In this article, we have analyzed the original statement and proofs of the reciprocity theorem proposed by Enrico Betti for a continuum body, first subjected to surface loading alone and then also in the presence of forces per unit mass. We have performed the exegesis of multiple passages of his treatise titled "Theory of elasticity" (*Teoria dell'elasticità*), dated 1872, enriched by the reference to some of his secondary works and to the treatise titled "Theory of Newtonian Forces and Its Applications to Electrostatics and Magnetism" (*La teorica delle forze che agiscono secondo la legge di Newton*), which was published in 1879 and also translated into German. We have tried to underline the theoretical (and historical) connections of these results with the potential theory and Green's identities, also discussing two exercises proposed by Betti and resting on this principle. Indeed, Betti was not able to generalize the reciprocity theorem in the presence of inelastic absolute displacements, prescribed over a part of the boundary of a solid body, or of inelastic relative displacements, commonly referred to as dislocations, subject that was investigated a few years later by Vito Volterra (1860–1940). The reciprocity principle, ultimately rooted in the potential theory and establishing huge analogies with Green's identities, represents an important step toward the integral identity of Carlo Somigliana (1860–1955), pupil of Enrico Betti and Eugenio Beltrami at the Scuola Normale Superiore of Pisa, that after some decades turned out to be essential for the development of the boundary element method.

In 1864, a few years before Betti's treatise on elasticity, J.C. Maxwell had proposed a method to solve statically indetermined frames, recognizing the symmetry of the influence coefficients relevant to displacements generated by unit actions. In 1877, Lord Rayleigh extended the formulation of Maxwell to vibrations and acoustics. In the meantime, the study of energetic principles applied to beams was pioneered by Luigi Federico Menabrea (1809–1896) and Carlo Alberto Castigliano (1847–1884). The treatise "Theorie der Elasticität fester Körper" published in 1862 by Alfred Clebsch (1833–1872) was translated into French in 1883 by A. De Saint-Venant (1797–1886), which enriched it with notes and comments. The application of the reciprocity principle to the elastic beams and frames was pushed forward by Robert Land (whose work [25] is dated 1887): in the spirit of the Technische Universität of those years, the dissemination of the theoretical knowledge was endowed by the rigorous attention for the engineering practice and the technological solutions, dominated by the figure of Einrich Müller Breslau (1851–1925). Other theoretical contributions on beams were provided by Vito Volterra [26] and Gustavo Colonnetti (1886–1968) [27,28]. In Italy, Giuseppe Albenga (1882–1957), an expert in bridges, got the merit to diffuse the influence lines due to traveling loads (see, e.g., Ref. [29]), which had an enormous impact on engineering design and practice and were further generalized, leading to influence surfaces for bidimensional plates. In this respect, the reciprocity principle allows one to deduce relationships, which often are neither intuitive nor trivial, involving displacements over the outer boundary and the external loading therein prescribed, but absolutely not including explicitly strains or stresses and not requiring the knowledge of the body's elastic stiffness. In other words, Betti's theorem allows one to observe a structure deforming when subjected to certain boundary conditions, and then to deduce its response under diverse boundary conditions, without solving the boundary value problem again. The reciprocity theorem can be regarded as an alternative formulation of the virtual work principle for structures behaving linearly elastically. It is worth noting that reciprocity theorems have also been formulated in other fields of physics and engineering, with reference to problems governed by self-adjoint operators, concerning as already mentioned the acoustics, also the electrodynamics, the electric currents in passive networks, and the hydraulics (see, e.g., Refs. [30–37]).

In light of the above critical discussion and historical analyses, it is worth emphasizing that several current research topics in solid mechanics can be placed in continuity with Betti's variational approach. In particular, the contemporary flourishing of nonlocal, higher-grade, or higher order models as well as of architected materials (see, e.g., Refs. [38–43]) can be regarded as a revival, under new mathematical and technological suggestions, of issues already present in the XIX century elasticity theory. Due to the discovery promoted by Germain [16], today it is well known that the elastic energy can be generalized up to include higher gradients of the displacement field or enriched kinematic descriptors (see, e.g., Refs. [44–47]). This assumption implies relaxing the principle of local action and renouncing to the paradigm of Cauchy (first gradient) continuum, specified by the dependence of the traction vector at a point exclusively on the second rank stress tensor and on the normal. During the last decades, the definition of such generalized energy densities has led to specify, through the principle of virtual work, exotic external actions not consistent with the Cauchy paradigm, such as edge and wedge forces, or generalized actions doing work on the normal derivatives of the virtual displacements (see, e.g., Refs. [48–50]). On one side, the presence of generalized boundary conditions demands for an improvement of the experimental activity through the design of nonconventional test configurations (see, e.g., Refs. [51–53]), suitable to identify the parameters governing such a constitutive relationship, in statics but especially in dynamics [54,55]. On the other side, some of these energies can be regarded as the macroscopic counterpart of peculiar microstructures that can be studied by homogenization or asymptotic methods, as developed, for instance, in Refs. [56–62]. Conventional mechanical problems can be given alternative formulations through high-order elastic models inducing regularization, see, e.g., Refs. [63,64]: bulk and surface high-order energy densities can be coupled consistently [65,66]. Besides the forward analyses, inverse strategies can also be envisaged with reference to such a typology of generalized materials: the parameters defining a class of microstructures can be regarded as the unknown variables to achieve prescribed properties at the macroscale, opening the way to synthesize novel meta-materials not existing in nature, see, e.g., Refs. [67–70]. These architected materials require often innovative fabrication processes (see, e.g., Ref. [71]) and, in the meantime, demands for alternative numerical strategies, implying an improvement of the currently available platforms for numerical simulations [72–74]. With reference to all the studies just mentioned, theoretical, numerical, and even experimental, we retain that the study of formulations leading to reciprocity as a form of symmetry, as well as the characterization of those scenarios in which such a reciprocity may be broken, can certainly contribute to further progress.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The authors attest that all data for this study are included in the article.

References

[1] Reale Accademia de' lincei, ed., 1903, *Opere Matematiche di Enrico Betti Volume I (1850–1860)*, Ulrico Hoepli, Milan.
[2] Gubernatis, A., 1879, *Dizionario Biografico degli scrittori contemporanei. Ornato con oltre 300 ritratti*, Le Monnier, Firenze.
[3] Calani, A., 1860, *Il Parlamento del Regno d'Italia: Opera illustrata dai ritratti degli onorevoli Senatori e Deputati*, Civelli, Milano.
[4] Perugi, G., 2023, "Sito della Scuola Normale Superiore di Pisa. Bicentenario della Nascita di Enrico Betti," Vita.
[5] Fedele, R., dell'Isola, F., and Barchiesi, E., 2026, *Encyclopedia of Continuum Mechanics*, 2nd ed., H. Altenbach, A. Öchsner, eds., Springer Nature, Berlin, (in print).
[6] Perugi, G., 2024, "Enrico Betti: l'impegno scientifico e civile di un matematico," Firenze Architettura, FUPress, **13**(2), pp. 17–28.
[7] Reale Accademia de' lincei, ed., 1913, *Opere Matematiche di Enrico Betti Volume II (1860–1890)*, Ulrico Hoepli, Milan.
[8] Capecchi, D., Ruta, G., and Tazzioli, R., 2006, *Enrico Betti: Teoria dell'elasticità*, Hevelius Edizioni, Benevento.
[9] Volterra, V., 1902, *Compte rendu du deuxième Congrès intern. des mathématiciens tenu à Paris du 6 au 12 Août 1900*, Enrique Duporcq, ed., Gauthier Villars, Paris, Aug. 6–12, pp. 43–57.
[10] Tazzioli, R., 2001, "Green's Function in Some Contributions of 19th Century Mathematicians," Historia Math., **28**(3), pp. 232–252.
[11] Fichera, G., 1979, "Il contributo italiano alla teoria matematica dell'elasticità," Rend. Circ. Mat. Palermo, **28**(Gennaio 1979), pp. 5–29.
[12] Pugno, G. M., 1952, *Leonardo da Vinci ed Enrico Betti*, Edizioni Ruata, Torino.
[13] Lebedev, L. P., Cloud, M. J., and Eremeyev, V. A., 2010, *Tensor Analysis With Applications in Mechanics*, World Scientific Publishing Company, London.

[14] Steigmann, D., and Shirani, M., 2025, *Principles of Continuum Mechanics*, World Scientific Publishing, New Jersey, US.
[15] dell'Isola, F., Maier, G., Perego, U., Andreaus, U., Esposito, R., and Forest, S., 2014, *The Complete Works of Gabrio Piola. Commented English Translation*, Vol. I, Springer International Publishing, Cham.
[16] Germain, P., 1973, "The Method of Virtual Power in the Mechanics of Continuous Media I: Second-Gradient Theory (English Translation, 2020)," Math. Mech. Complex Syst., **8**(2), pp. 153–190.
[17] Eugster, R. S., 2017, "Exegesis of the Introduction and Sect. I From "Fundamentals of the Mechanics of Continua" by E. Hellinger," Z. Angew. Math. Mech., **97**(4), pp. 477–506.
[18] Maxwell, J. C., 1864, "On the Calculation of the Equilibrium and Stiffness of Frames," Lond. Edinb. Dubl. Philos. Mag. J. Sci., **28**, pp. 294–299.
[19] Strutt, J. W., and Rayleigh, B., 1877, *The Theory of Sound*, Vol. I, MacMillan and Co, London.
[20] Benvenuto, E., 2001, *An Introduction to the History of Structural Mechanics. Part I: Statics and Resistance of Solids*, Springer Verlag, New York.
[21] Russo, L., 2004, *The Forgotten Revolution: How Science Was Born in 300 BC and Why It Had to Be Reborn*, Springer, GmbH, Berlin.
[22] Spagnuolo, M., 2025, "Early Traces of the Principle of Virtual Work in Hellenistic Texts: Exegesis of Problems 1, 2 and 3 in Pseudo-Aristotle's Mechanica Problemata," Mech. Res. Commun., **148**, p. 104499.
[23] Kellogg, O. D., 1929, *Foundations of Potential Theory*, Frederick Ungar Publishing Company, New York.
[24] Gilbarg, D., and Trudinger, N. S., 2001, Elliptic Partial Differential Equations of Second Order *(Classics in Mathematics)*, Springer, Berlin.
[25] Land, R., 1887, "Die Gegenseitigkeit elastischer Formänderung als Grundlage einer allgemeinen Darstellung der Einflusslinien aller Trägerarten, Wochenblatt für Baukunde."
[26] Volterra, V., 1907, "Sur l'équilibre des corps élastiques multiplement connexes," Ann. Sci. Éc. Norm. Supér., **XXIV**, pp. 401–517.
[27] Colonnetti, G., 1928, *La statica delle costruzioni*, Unione Tipografica-Editrice Torinese, Torino.
[28] Bonvicini, D., 1931, "Sopra una formula comprendente tutti i teoremi di reciprocità della statica dei solidi elastici," Bollettino del Sindacato Provinciale Fascista Ingegneri di Bologna.
[29] Albenga, G., 1915, "Sul teorema di reciprocità di Land," Atti della R. Accademia delle Scienze di Torino, **L**.
[30] Lelli, M., 1926, "Il principio di reciprocità nella Fisica," Annali di Matematica, **3**(1), pp. 133–170.
[31] Truesdell, C., 1963, "The Meaning of Betti's Reciprocal Theorem," J. Res. Nat. Bureau Stand. Sec. B: Math. Math. Phys., **67B**(2), pp. 85–86.
[32] Cook, R. K., 1996, "Lord Rayleigh and Reciprocity in Physics," J. Acoust. Soc. Am., **99**(1), pp. 24–29.
[33] Achenbach, J. D., 2006, "Reciprocity and Related Topics in Elastodynamics," ASME Appl. Mech. Rev., **59**(1), pp. 13–32.
[34] Mansuripur, M., and Tsai, D., 2011, "New Perspective on the Reciprocity Theorem of Classical Electrodynamics," Opt. Commun., **284**(3), pp. 707–714.
[35] Samarasinghe, P., Abhayapala, T. D., and Kellermann, W., 2017, "Acoustic Reciprocity: An Extension to Spherical Harmonics Domain," J. Acoust. Soc. Am., **142**(4), pp. EL337–EL343.
[36] Nassar, H., Yousefzadeh, B., Fleury, R., Ruzzene, M., Alù, A., Daraio, C., and Norris, A. N., 2020, "Nonreciprocity in Acoustic and Elastic Materials," Nat. Rev. Mater., **5**, pp. 667–685.
[37] Nakshatrala, K. B., 2024, "Dynamic Properties of Double Porosity/Permeability Model," ASME J. Appl. Mech., **91**(6), p. 061005.
[38] dell'Isola, F., Andreaus, U., and Placidi, L., 2015, "At the Origins and in the Vanguard of Peridynamics, Non-Local and Higher-Gradient Continuum Mechanics: An Underestimated and Still Topical Contribution of Gabrio Piola," Math. Mech. Solids, **20**(8), pp. 887–928.
[39] dell'Isola, F., Della Corte, A., and Giorgio, I., 2016, "Higher-Gradient Continua: The Legacy of Piola, Mindlin, Sedov and Toupin and Some Future Research Perspectives," Math. Mech. Solids, **22**(4), pp. 852–872.
[40] Apuzzo, A., Barretta, R., Luciano, R., de Sciarra, F. M., and Penna, R., 2017, "Free Vibrations of Bernoulli-Euler Nano-beams by the Stress-Driven Nonlocal Integral Model," Compos. Part B Eng., **123**, pp. 105–111.
[41] Barretta, R., Čanađija, M., Luciano, R., and de Sciarra, F. M., 2018, "Stress-Driven Modeling of Nonlocal Thermoelastic Behavior of Nanobeams," Int. J. Eng. Sci., **126**, pp. 53–67.
[42] Barretta, R., Čanadija, M., Feo, L., Luciano, R., de Sciarra, F. M., and Penna, R., 2018, "Exact Solutions of Inflected Functionally Graded Nano-Beams in Integral Elasticity," Compos. Part B Eng., **142**, pp. 273–286.
[43] McAvoy, R. C., and Shirani, M., 2025, "Asymptotic Theory for Thin Nonlinear Second-Gradient Elastic Plates," Math. Mech. Complex Syst., **13**(4), pp. 391–416.
[44] Eremeyev, V. A., and Pietraszkiewicz, W., 2016, "Material Symmetry Group and Constitutive Equations of Micropolar Anisotropic Elastic Solids," Math. Mech. Solids, **21**(2), pp. 210–221.
[45] Misra, A., Placidi, L., and Barchiesi, E., 2021, "Identification of a Geometrically Nonlinear Micromorphic Continuum Via Granular Micromechanics," Z. Angew. Math. Phys., **72**(4), p. 157.
[46] Desmorat, B., 2025, "Energy Theorems for Micromorphic Media," Math. Mech. Complex Syst., **14**(1), pp. 1–10.
[47] La Valle, G., Soize, C., and Steigmann, D. J., 2025, "A Cauchy–Piola Framework for Micro-Based Micromorphic Continua," Math. Mech. Complex Syst., **14**(1), pp. 71–94.
[48] Fedele, R., 2022, "Deformation Induced Coupling of the Generalized External Actions in Third-Gradient Materials," Z. Angew. Math. Phys., **73**(28 Sept 2022), p. 218.
[49] dell'Isola, F., and Fedele, R., 2023, "Irreducible Representation of Surface Distributions and Piola Transformation of External Loads Sustainable by Third Gradient Continua," C. R. Mecanique, **351**, pp. 1–30.
[50] Fedele, R., 2025, "Eulerian–Lagrangian Transformation of H-Forces in Generalized Elasticity by Recursive Projection and Integration," Math. Mech. Solids.
[51] Auger, P., Lavigne, T., Smaniotto, B., Spagnuolo, M., and Hild, F., 2020, "Poynting Effects in Pantographic Metamaterial Captured Via Multiscale DVC," J. Strain Anal. Eng. Des., **56**(7), pp. 462–477.
[52] Spagnuolo, M., Andreaus, U., Misra, A., Giorgio, I., and Hild, F., 2021, "Mesoscale Modeling and Experimental Analyses for Pantographic Cells: Effect of Hinge Deformation," Mech. Mater., **160**, p. 103924.
[53] Ciallella, A., Murcia Terranova, L., Smaniotto, B., Vintache, A., and Hild, F., 2025, "Model-Driven Digital Volume Correlation: A Step Forward in Experimental Analyses of Metamaterial Deformations," Math. Mech. Solids.
[54] Placidi, L., de Castro Motta, J., and Fraternali, F., 2024, "Bandgap Structure of Tensegrity Mass–Spring Chains Equipped With Internal Resonators," Mech. Res. Commun., **137**, p. 104273.
[55] De Fazio, N., Placidi, L., Fabbrocino, F., and Luciano, R., 2025, "Analysis of Ultrasonic Wave Dispersion in Presence of Attenuation and Second-Gradient Contributions," Civil Eng., **6**(3), p. 37.
[56] Rahali, Y., Giorgio, I., and Ganghoffer, J. F., 2015, "Homogenization à la Piola Produces Second Gradient Continuum Models for Linear Pantographic Lattices," Int. J. Eng. Sci., **97**, pp. 148–172.
[57] Carcaterra, A., dell'Isola, F., Esposito, R., and Pulvirenti, M., 2015, "Macroscopic Description of Microscopically Strongly Inhomogenous Systems: A Mathematical Basis for the Synthesis of Higher Gradients Metamaterials," Arch. Ration. Mech. Anal., **218**(3), pp. 1239–1262.
[58] Boutin, C., Giorgio, I., and Placidi, L., 2017, "Linear Pantographic Sheets: Asymptotic Micro-Macro Models Identification," Math. Mech. Complex Syst., **5**(2), pp. 127–162.
[59] dell'Isola, F., Seppecher, P., Alibert, J. J., Lekszycki, T., Grygoruk, R., Pawlikowski, M., Steigmann, D., et al., 2019, "Pantographic Metamaterials: An Example of Mathematically Driven Design and of Its Technological Challenges," Contin. Mech. Thermodyn., **31**(4), pp. 851–884.
[60] Yang, H., Abali, B. E., Timofeev, D., and Müller, W. H., 2020, "Determination of Metamaterial Parameters by Means of a Homogenization Approach Based on Asymptotic Analysis," Contin. Mech. Thermodyn., **32**, pp. 1251–1270.
[61] Eremeyev, V. A., and Berinskii, I., 2026, "Positive Definiteness of Deformation Energy as a Consistency Condition of Discrete-to-Continuum Transition," Contin. Mech. Thermodyn., **38**, p. 2.
[62] Rizzi, N. L., and Seppecher, P., 2026, "A Third-Gradient 1D Continuum Obtained Via Asymptotic Expansion From a Micro-Structure Obtained Constraining a Sequence of Modified Hart's Antiparallelograms," Math. Mech. Solids.
[63] Xie, J., Javili, A., and Linder, C., 2025, "The Kirsch Problem in Second Strain Gradient Elasticity," Proc. A, **481**(2139), p. 20250389.
[64] Ulukhanyan, A., Placidi, L., Fedele, R., Misra, A., Barchiesi, E., and Fabbrocino, F., 2025, "Embedding 1D Euler Beam in 2D Second-Gradient Continua," Math. Mech. Solids.
[65] Javili, A., Dell'Isola, F., and Steinmann, P., 2013, "Geometrically Nonlinear Higher-Gradient Elasticity With Energetic Boundaries," J. Mech. Phys. Solids, **61**(12), pp. 2381–2401.
[66] Eremeyev, V. A., 2016, "On Effective Properties of Materials at the Nano- and Microscales Considering Surface Effects," Acta Mech., **227**(1), pp. 29–42.
[67] Giorgio, I., 2016, "Numerical Identification Procedure Between a Micro-Cauchy Model and a Macro-second Gradient Model for Planar Pantographic Structures," Z. Angew. Math. Phys., **67**(4), p. 95.
[68] Turco, E., Giorgio, I., and Misra, A., 2017, "King Post Truss as a Motif for Internal Structure of (Meta) Material With Controlled Elastic Properties," R. Soc. Open Sci., **4**(10), p. 171153.
[69] Giorgio, I., 2021, "Lattice Shells Composed of Two Families of Curved Kirchhoff Rods: An Archetypal Example, Topology Optimization of a Cycloidal Metamaterial," Contin. Mech. Thermodyn., **33**(4), pp. 1063–1082.
[70] De Angelo, M., Barchiesi, E., Giorgio, I., and Abali, B. E., 2019, "Numerical Identification of Constitutive Parameters in Reduced-Order Bi-dimensional Models for Pantographic Structures: Application to Out-of-Plane Buckling," Arch. Appl. Mech., **89**(7), pp. 1333–1358.
[71] Sanna, G., Pistis, M., Giorgio, I., Eremeyev, V. A., and Spagnuolo, M., 2025, "New Hinge Design for Fibrous Metamaterial Enables for Filament 3D Printing," Mech. Res. Commun., **148**, p. 104452.
[72] Sarar, B. C., Yildizdag, M. E., Fabbrocino, F., and Abali, B. E., 2025, "A Comparative Analysis for Different Finite Element Types in Strain-Gradient Elasticity Simulations Performed on Firedrake and FEniCS," Math. Mech. Complex Syst., **13**(3), pp. 237–252.
[73] Abali, B. E., Müller, W. H., and Eremeyev, V. A., 2015, "Strain Gradient Elasticity With Geometric Nonlinearities and Its Computational Evaluation," Mech. Adv. Mater. Mod. Process., **1**, p. 4.
[74] Succi, S., 2025, "On the Physical Interpretation of Neural PDEs," Math. Mech. Complex Syst., **13**(3), pp. 275–286.